Product Churn and the GEKS-Törnqvist Price Index: The "Feenstra Adjustment"

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Abstract: Several statistical agencies are using the multilateral GEKS-Törnqvist method to construct sub-components of the CPI from scanner data, particularly for groceries. This paper examines the relationship between the GEKS-Törnqvist price index and the CES (Constant Elasticity of Substitution) cost-of-living index. We first discuss the unrealistic case with no product churn and show that, under CES assumptions, the substitution bias in the GEKS-Törnqvist index against the CES index depends on the extent to which the GEKS-Törnqvist violates the product test, given the value of the elasticity of substitution. Next, we outline the "Feenstra adjustment" to account for new and disappearing goods. We also describe two methods for estimating the elasticity of substitution, an algebraic method that is based on matched-model Törnqvist and Jevons price and quantity indexes and a regression method. The effect of the Feenstra adjustment on GEKS-Törnqvist price indexes is illustrated for a range of product categories using IRI scanner data. The paper concludes with a discussion of potential problems.

Keywords: CES cost-of-living index, multilateral methods, new and disappearing goods, scanner data.

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1. Introduction

Statistical agencies are increasingly using multilateral price index methods to construct CPI sub-components from barcode scanning data, in particular GEKS (Gini, 1931; Eltetö and Köves, 1964; Szulc, 1964) with matched-model bilateral Törnqvist price indexes as elements (Van Kints, De Haan and Webster, 2020).[1] Multilateral indexes are transitive and hence free of chain drift. Ivancic, Diewert and Fox (2011) proposed using multilateral methods to deal with potential chain drift in weighted price indexes due to sales.[2]

Matched-model price indexes do not account for new and disappearing products. Product churn in scanner data is often significant, and ignoring unmatched products could bias the results. Imputing the "missing prices" of the unmatched products using hedonic regression explicitly adjusts the GEKS-Törnqvist price index for quality change (De Haan and Krsinich, 2014; De Haan and Daalmans, 2019). However, hedonic regressions need information on price-determining characteristics for all products. Omitted variables bias will result if important characteristics cannot be observed.

Assuming consumer preferences are represented by the CES (Constant Elasticity of Substitution) utility function, Feenstra (1994) proposed a method that does not rely on product characteristics; it uses the matched products' expenditure shares and the value of the elasticity of substitution to adjust the CES cost-of-living index. The Sato-Vartia (Sato, 1976; Vartia, 1976) price index is exact for CES preferences. So, if we have an estimate of the elasticity, his solution can be readily applied to scanner data.[3]

We address a couple of issues with Feenstra's proposal. While the CES index and therefore, under CES assumptions, the Sato-Vartia price index are transitive, disturbances may lead to non-transitivity when applied to real-world data. Even without disturbances, Feenstra's adjustment produces an index that is not necessarily transitive. Also, statistical agencies prefer the superlative Törnqvist price index to the Sato-Vartia. To resolve these issues, we derive an expression for the adjusted CES index in terms of the Törnqvist price index and then apply the GEKS procedure to impose transitivity.

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[1] The GEKS-Törnqvist index is also known as CCDI (Caves, Christensen and Diewert, 1982; Inklaar and Diewert, 2016) index. Since we also discuss the GEKS-Jevons index, which has no synonym, we decided to use the first name.

[2] Diewert (2022) provides a thorough discussion of the chain drift problem.

[3] For early applications of the Feenstra adjustment to scanner data, see Opperdoes (2001), De Haan (2002; 2005), and Melser (2006).

The paper is organized as follows. Section 2 recalls the definition of the CES cost-of-living index and some established facts. Section 3 discusses two representations of the CES index: the product of the Törnqvist price index, or the Jevons price index, and a term that depends on the extent to which the index violates the product test, given the value of the elasticity of substitution. Section 4 applies GEKS to the Törnqvist CES representation and provides an expression for the (substitution) bias of the GEKS-Törnqvist price index against the CES price index. The Feenstra adjustment then is applied to account for new and disappearing products and defines the adjusted GEKS-Törnqvist price index. Section 5 outlines two methods for estimating the elasticity of substitution: an algebraic method using matched-model Törnqvist and Jevons price and quantity indexes, and a regression method. Section 6 describes the IRI scanner data we utilize and presents empirical results. Section 7 discusses some conceptual and practical problems with the CES index and the Feenstra adjustment.

2. Some established facts

To summarize some established facts about the CES cost-of-living index, in this section (and the next) we assume that there is no product churn; the set of products purchased by consumers, denoted by *I* with size *N*, is fixed. The elasticity of substitution, σ, measures how the quantities demanded in period *t* for products $i \in I$ and $j \in I$, x_i^t and x_j^t, depend on the (relative) prices, p_i^t and p_j^t. It is defined as

$$\sigma = -\frac{\partial \ln(x_i^t / x_j^t)}{\partial \ln(p_i^t / p_j^t)} \; (\sigma \geq 0). \tag{1}$$

The CES utility function assumes that the elasticity of substitution is the same for all pairs of products and fixed across time.

The corresponding CES cost-of-living index, or CES (price) index as we will refer to it, going from the base period 0 to period *t* $(t = 0,...,T)$ is given by[4]

$$P_{CES}^{0t} = \left[\frac{\sum_{i \in I} b_i (p_i^t)^{1-\sigma}}{\sum_{i \in I} b_i (p_i^0)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}. \tag{2}$$

[4] CES preferences are homothetic so that the CES price index is independent of the utility level.

Equation (2) is not defined for $\sigma = 1$; for $\sigma \to 1$ the CES price index equals the Törnqvist price index. The b_i are taste parameters. A (conventional) cost-of-living index measures the effect of price change only, and so the taste parameters are kept constant across time. The CES index is transitive and can be written in period-on-period chained form.

By Shephard's Lemma, the period *t* expenditure shares, $s_i^t = p_i^t x_i^t / \sum_{i \in I} p_i^t x_i^t$, are equal to

$$s_i^t = \frac{b_i (p_i^t)^{1-\sigma}}{\sum_{i \in I} b_i (p_i^t)^{1-\sigma}}. \tag{3}$$

Dividing the period *t* expenditure share (3) for product *i* by the period 0 share yields

$$\frac{s_i^t}{s_i^0} = \left(\frac{p_i^t}{p_i^0}\right)^{1-\sigma} \left[\frac{\sum_{i \in I} b_i (p_i^t)^{1-\sigma}}{\sum_{i \in I} b_i (p_i^0)^{1-\sigma}}\right]^{-1}. \tag{4}$$

It follows that

$$\left(\frac{s_i^t}{s_i^0}\right)^{\frac{1}{\sigma-1}} \frac{p_i^t}{p_i^0} = \left[\frac{\sum_{i \in I} b_i (p_i^t)^{1-\sigma}}{\sum_{i \in I} b_i (p_i^0)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}} = P_{CES}^{0t}. \tag{5}$$

A few interesting points emerge from equation (5). First, if the CES assumptions are satisfied, we would only require data on prices and expenditure shares in the periods compared for a single product, plus the value of σ, to be able to calculate the CES index. Second, it is easy to verify that the left-hand side of (5) for two products, *i* and *j*, leads to $\sigma = 1 - \ln[(s_i^t / s_i^0)/(s_j^t / s_j^0)] / \ln[(p_i^t / p_i^0)/(p_j^t / p_j^0)]$, showing that prices and expenditure shares for only two products in two periods suffice to calculate σ. Third, for $\sigma \to \infty$ the CES index simplifies to p_i^t / p_i^0. This result is not surprising as we expect price changes of perfect substitutes to be the same.

3. Two geometric representations of the CES index

We focus here on the following implication of equation (5): any average of the left-hand side across all products will be equal to the CES index. This means there is a whole range – actually an infinite number – of different representations of the CES index, depending on the type of average and the weights used; see also Banerjee (1983). For example, we could take an arithmetic, geometric or harmonic average of the left-hand side of (5) across all products. The averages may be unweighted or weighted, and the weights a_i^t may be time-dependent, provided that they satisfy $\sum_{i \in I} a_i^t = 1$ in every period. That is, the CES index can be written as[5]

$$P_{CES}^{0t} = \sum_{i \in I} a_i^t \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{\sigma-1}} \left(\frac{p_i^t}{p_i^0} \right) = \prod_{i \in I} \left(\frac{s_i^t}{s_i^0} \right)^{\frac{a_i^t}{\sigma-1}} \left(\frac{p_i^t}{p_i^0} \right)^{a_i^t} = \left[\sum_{i \in I} a_i^t \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{1-\sigma}} \left(\frac{p_i^t}{p_i^0} \right)^{-1} \right]^{-1}. \qquad (6)$$

We are assuming here that all types of averages are equally valid, which is tantamount to assuming that prices are measured without error and the expenditure shares given by (3) hold exactly. The last assumption is of course very strong.[6]

For our purpose, two geometric representations are important, an unweighted one and a weighted one. Setting $a_i^t = 1/N$ in the second expression of (6) yields

$$P_{CES}^{0t} = \left[\prod_{i \in I} \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{\sigma-1}} \right]^{\frac{1}{N}} \prod_{i \in I} \left(\frac{p_i^t}{p_i^0} \right)^{\frac{1}{N}} = \left[\frac{\prod_{i \in I} (s_i^t)^{\frac{1}{N}}}{\prod_{i \in I} (s_i^0)^{\frac{1}{N}}} \right]^{\frac{1}{\sigma-1}} P_J^{0t}, \qquad (7)$$

where $P_J^{0t} = \prod_{i \in I} (p_i^t / p_i^0)^{1/N}$ is the Jevons price index. This representation is Redding and Weinstein's (2020) "CES Common Varieties Index" although they derived the result in a different way.

Setting $a_i^t = (s_i^0 + s_i^t)/2$ in the second expression of (6) yields

$$P_{CES}^{0t} = \prod_{i \in I} \left(\frac{s_i^t}{s_i^0} \right)^{\frac{s_i^0 + s_i^t}{2(\sigma-1)}} P_T^{0t} = \left[\frac{\prod_{i \in I} (s_i^t)^{\frac{s_i^0 + s_i^t}{2}}}{\prod_{i \in I} (s_i^0)^{\frac{s_i^0 + s_i^t}{2}}} \right]^{\frac{1}{\sigma-1}} P_T^{0t}, \qquad (8)$$

where $P_T^{0t} = \prod_{i \in I} (p_i^t / p_i^0)^{(s_i^0 + s_i^t)/2}$ is the Törnqvist price index. This representation of the CES index was independently derived by De Haan (2019) and Kurtzon (2020).

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[5] Appendix 1 discusses expressions for the CES index for some values of the elasticity.

[6] Different types of averages may be "best" for different error distributions. For example, if the left-hand side of equation (5) had additive random error with constant variance, the unweighted arithmetic average across all products might be the preferred choice. See also Section 5 where we discuss a linear regression model based on equation (5) for estimating the elasticity of substitution.

Using $s_i^0 = p_i^0 x_i^0 / \sum_{i \in I} p_i^0 x_i^0$ and $s_i^t = p_i^t x_i^t / \sum_{i \in I} p_i^t x_i^t$, the bracketed term in (7) can be written as

$$\frac{\prod_{i \in I} (s_i^t)^{\frac{1}{N}}}{\prod_{i \in I} (s_i^0)^{\frac{1}{N}}} = \frac{\dfrac{\prod_{i \in I} (p_i^t x_i^t)^{\frac{1}{N}}}{\prod_{i \in I} \left[\sum_{i \in I} p_i^t x_i^t \right]^{\frac{1}{N}}}}{\dfrac{\prod_{i \in I} (p_i^0 x_i^0)^{\frac{1}{N}}}{\prod_{i \in I} \left[\sum_{i \in I} p_i^0 x_i^0 \right]^{\frac{1}{N}}}} = \frac{\dfrac{\prod_{i \in I} (p_i^t x_i^t)^{\frac{1}{N}}}{\prod_{i \in I} (p_i^0 x_i^0)^{\frac{1}{N}}}}{\dfrac{\sum_{i \in I} p_i^t x_i^t}{\sum_{i \in I} p_i^0 x_i^0}} = \frac{\prod_{i \in I} \left(\dfrac{p_i^t}{p_i^0} \right)^{\frac{1}{N}} \prod_{i \in I} \left(\dfrac{x_i^t}{x_i^0} \right)^{\frac{1}{N}}}{\dfrac{\sum_{i \in I} p_i^t x_i^t}{\sum_{i \in I} p_i^0 x_i^0}} . \tag{9}$$

We denote the aggregate values in period 0 and period *t*, $\sum_{i \in I} p_i^0 x_i^0$ and $\sum_{i \in I} p_i^t x_i^t$, by V^0 and V^t, and refer to the unweighted geometric quantity index $Q_J^{0t} = \prod_{i \in I} (x_i^t / x_i^0)^{1/N}$ as the Jevons quantity index. A shorthand notation for (9) therefore is

$$\frac{\prod_{i \in I} (s_i^t)^{\frac{1}{N}}}{\prod_{i \in I} (s_i^0)^{\frac{1}{N}}} = \frac{P_J^{0t} Q_J^{0t}}{V^t / V^0} . \tag{10}$$

The Jevons formula does not satisfy the product test, i.e., in general, $P_J^{0t} Q_J^0 \neq V^t / V^0$, so that (10) will not be equal to 1. Substituting (10) into (7) gives

$$P_{CES}^{0t} = \left[\frac{P_J^{0t} Q_J^{0t}}{V^t / V^0} \right]^{\frac{1}{\sigma - 1}} P_J^{0t} . \tag{11}$$

For equation (8) we follow similar steps. The term between square brackets in (8) can be written as

$$\frac{\prod_{i \in I} (s_i^t)^{\frac{s_i^0 + s_i^t}{2}}}{\prod_{i \in I} (s_i^0)^{\frac{s_i^0 + s_i^t}{2}}} = \frac{\dfrac{\prod_{i \in I} (p_i^t x_i^t)^{\frac{s_i^0 + s_i^t}{2}}}{\prod_{i \in I} \left[\sum_{i \in I} p_i^t x_i^t \right]^{\frac{s_i^0 + s_i^t}{2}}}}{\dfrac{\prod_{i \in I} (p_i^0 x_i^0)^{\frac{s_i^0 + s_i^t}{2}}}{\prod_{i \in I} \left[\sum_{i \in I} p_i^0 x_i^0 \right]^{\frac{s_i^0 + s_i^t}{2}}}} = \frac{\dfrac{\prod_{i \in I} (p_i^t x_i^t)^{\frac{s_i^0 + s_i^t}{2}}}{\prod_{i \in I} (p_i^0 x_i^0)^{\frac{s_i^0 + s_i^t}{2}}}}{\dfrac{\sum_{i \in I} p_i^t x_i^t}{\sum_{i \in I} p_i^0 x_i^0}} = \frac{\prod_{i \in I} \left(\dfrac{p_i^t}{p_i^0} \right)^{\frac{s_i^0 + s_i^t}{2}} \prod_{i \in I} \left(\dfrac{x_i^t}{x_i^0} \right)^{\frac{s_i^0 + s_i^t}{2}}}{\dfrac{\sum_{i \in I} p_i^t x_i^t}{\sum_{i \in I} p_i^0 x_i^0}} . \tag{12}$$

A shorthand notation for (12) is

$$\frac{\prod_{i\in I}(s_i^t)^{\frac{s_i^0+s_i^t}{2}}}{\prod_{i\in I}(s_i^0)^{\frac{s_i^0+s_i^t}{2}}}=\frac{P_T^{0t}Q_T^{0t}}{V^t/V^0}, \tag{13}$$

where $\prod_{i\in I}(x_i^t/x_i^0)^{(s_i^0+s_i^t)/2}=Q_T^{0t}$ is the Törnqvist quantity index. The Törnqvist formula does not satisfy the product test either. Substituting (13) into (8) gives

$$P_{CES}^{0t}=\left[\frac{P_T^{0t}Q_T^{0t}}{V^t/V^0}\right]^{\frac{1}{\sigma-1}}P_T^{0t}. \tag{14}$$

4. GEKS-Törnqvist and the Feenstra adjustment

The GEKS price index between periods 0 and *t* is calculated as the unweighted geometric average of the ratios of the bilateral price indexes P^{lt} and P^{l0}, which should satisfy the time reversal test, across the entire sample period $0,...,T$, where each period *l* is taken as the base. That is, the GEKS index is given by (De Haan and Van der Grient, 2011)

$$P_{GEKS}^{0t}=\prod_{l=0}^{T}\left[\frac{P^{lt}}{P^{l0}}\right]^{\frac{1}{T+1}}=\prod_{l=0}^{T}\left[P^{0l}\times P^{lt}\right]^{\frac{1}{T+1}}. \tag{15}$$

The GEKS index, like any multilateral index, is transitive; it is independent of the choice of base period (in our case period 0) and can be written as a chained index. Because the CES index itself is transitive and satisfies the time reversal test, it can be written as a GEKS-CES index:

$$P_{CES}^{0t}=P_{GEKSCES}^{0t}=\prod_{l=0}^{T}\left[P_{CES}^{0l}\times P_{CES}^{lt}\right]^{\frac{1}{T+1}}. \tag{16}$$

From equation (14) we know that the CES index between period 0 and period *l* (*l*=0,…,*T*) in (16) can be written as

$$P_{CES}^{0l}=\left[\frac{P_T^{0l}Q_T^{0l}}{V^l/V^0}\right]^{\frac{1}{\sigma-1}}P_T^{0l}. \tag{17}$$

Substituting (17) into (16), and similarly for the CES index between periods *l* and *t*, gives

$$P_{CES}^{0t}=\prod_{l=0}^{T}\left[\frac{P_T^{0l}Q_T^{0l}}{V^l/V^0}\frac{P_T^{lt}Q_T^{lt}}{V^t/V^l}\right]^{\frac{1}{\sigma-1}}P_{GEKST}^{0t}=\left[\frac{P_{GEKST}^{0t}Q_{GEKST}^{0t}}{V^t/V^0}\right]^{\frac{1}{\sigma-1}}P_{GEKST}^{0t}, \qquad (18)$$

where $P_{GEKST}^{0t}=\prod_{l=0}^{T}\left[P_T^{0l}\times P_T^{lt}\right]^{1(T+1)}$ and $Q_{GEKST}^{0t}=\prod_{l=0}^{T}\left[Q_T^{0l}\times Q_T^{lt}\right]^{1(T+1)}$ are the GEKS-Törnqvist price and quantity indexes.

The difference between the GEKS-Törnqvist price index and the CES price index depends on the extent to which GEKS-Törnqvist violates the product test, given the value of the elasticity of substitution. We can view

$$\frac{P_{GEKST}^{0t}}{P_{CES}^{0t}}=\left[\frac{P_{GEKST}^{0t}Q_{GEKST}^{0t}}{V^t/V^0}\right]^{\frac{1}{1-\sigma}} \qquad (19)$$

as a measure of substitution bias in the GEKS-Törnqvist price index against the CES price index.[7]

Assuming no product churn, as was done so far, is of course unrealistic. Feenstra (1994) proposed a very simple adjustment to account for new and disappearing products. We denote the sets of products purchased in periods 0 and t $(t=1,...,T)$ by I^0 and I^t, and the set of matched products, $I^0\cap I^t$, by $I_{M(0t)}$. The matched-model CES index going from period 0 to period t is given by

$$P_{MCES}^{0t}=\left[\frac{\sum_{i\in I_{M(0t)}}b_i(p_i^t)^{1-\sigma}}{\sum_{i\in I_{M(0t)}}b_i(p_i^0)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}. \qquad (20)$$

The adjusted CES price index is defined as

$$P_{CES(A)}^{0t}=\left[\frac{V_{M(0t)}^t/V^t}{V_{M(0t)}^0/V^0}\right]^{\frac{1}{\sigma-1}}P_{MCES}^{0t}, \qquad (21)$$

where $V_{M(0t)}^0=\sum_{i\in I_{M(0t)}}p_i^0x_i^0$, $V_{M(0t)}^t=\sum_{i\in I_{M(0t)}}p_i^tx_i^t$, $V^0=\sum_{i\in I^0}p_i^0x_i^0$ and $V^t=\sum_{i\in I^t}p_i^tx_i^t$. Thus, the Feenstra adjustment term equals the ratio of the matched expenditure shares in the periods compared, $V_{M(0t)}^t/V^t$ and $V_{M(0t)}^0/V^0$, raised to the power $1/(\sigma-1)$.

[7] Using unit value prices from scanner data, Diewert and Fox (2018), and also Melser and Webster (2019), calculated the products' expenditure shares implied by CES preferences for different values of the elasticity of substitution and then compared the resulting CES indexes with various multilateral price indexes based on the same data. They did not calculate substitution bias in the GEKS-Törnqvist price index according to (19), but their simulations conformed that the bias was on average quite small.

The matched-model counterpart to representation (14) of the CES price index is

$$P_{MCES}^{0t} = \left[\frac{P_{MT}^{0t} Q_{MT}^{0t}}{V_{M(0t)}^{t} \big/ V_{M(0t)}^{0}} \right]^{\frac{1}{\sigma-1}} P_{MT}^{0t}, \tag{22}$$

where $P_{MT}^{0t} = \prod_{i \in I_{M(0t)}} (p_i^t / p_i^0)^{(s_i^0/\sum_{i\in I_{M(0t)}} s_i^0 + s_i^t/\sum_{i\in I_{M(0t)}} s_i^t)/2}$ is the matched-model Törnqvist price index and $Q_{MT}^{0t} = \prod_{i \in I_{M(0t)}} (x_i^t / x_i^0)^{(s_i^0/\sum_{i\in I_{M(0t)}} s_i^0 + s_i^t/\sum_{i\in I_{M(0t)}} s_i^t)/2}$ is the matched-model Törnqvist quantity index. Using (22), expression (21) for the adjusted CES index becomes

$$P_{CES(A)}^{0t} = \left[\frac{P_{MT}^{0t} Q_{MT}^{0t}}{V^t / V^0} \right]^{\frac{1}{\sigma-1}} P_{MT}^{0t} = \left[\frac{Q_{MT}^{0t}}{(V^t / V^0) / P_{MT}^{0t}} \right]^{\frac{1}{\sigma-1}} P_{MT}^{0t}. \tag{23}$$

The denominator in the bracketed term of the second expression, $(V^t / V^0) / P_{MT}^{0t}$, deflates the aggregate value ratio by the matched-model Törnqvist price index and can be viewed as an implicit quantity index that includes new and disappearing products.

In contrast to the CES index, the adjusted CES index is not necessarily transitive. Transitivity can be achieved by applying the GEKS procedure to equation (23), yielding the adjusted GEKS-CES index:

$$P_{GEKSCES(A)}^{0t} = \left[\frac{P_{GEKSMT}^{0t} Q_{GEKSMT}^{0t}}{V^t / V^0} \right]^{\frac{1}{\sigma-1}} P_{GEKSMT}^{0t} = \left[\frac{Q_{GEKSMT}^{0t}}{(V^t / V^0) / P_{GEKSMT}^{0t}} \right]^{\frac{1}{\sigma-1}} P_{GEKSMT}^{0t}, \tag{24}$$

where $P_{GEKSMT}^{0t} = \prod_{l=0}^{T} \left[P_{MT}^{0l} \times P_{MT}^{lt} \right]^{1(T+1)}$ and $Q_{GEKSMT}^{0t} = \prod_{l=0}^{T} \left[Q_{MT}^{0l} \times Q_{MT}^{lt} \right]^{1(T+1)}$ are GEKS-Törnqvist price and quantity indexes based on the matched-model bilateral comparisons. $(V^t / V^0) / P_{GEKSMT}^{0t}$ is again an implicit quantity index with the GEKS-Törnqvist price index now being the deflator.

$P_{GEKSCES(A)}^{0t}$ can be viewed as the "optimal" deflator to be used in, e.g., the National Accounts, and the corresponding quantity index is $Q^{0t} = (V^t / V^0) / P_{GEKSCES(A)}^{0t}$. Substituting $V^t / V^0 = P_{GEKSCES(A)}^{0t} Q^{0t}$ into (24) and solving for $P_{GEKSCES(A)}^{0t}$ yields (assuming $\sigma > 0$)

$$P_{GEKSCES(A)}^{0t} = \left[\frac{Q_{GEKSMT}^{0t}}{Q^{0t}} \right]^{\frac{1}{\sigma}} P_{GEKSMT}^{0t}. \tag{25}$$

Equation (25) is not very helpful from a practical perspective, but it does provide some additional insight as it compares Q_{GEKSMT}^{0t} with the optimal quantity index Q^{0t}. Note that if P_{GEKSMT}^{0t} is upward biased against $P_{GEKSCES(A)}^{0t}$, then Q_{GEKSMT}^{0t} must be downward biased against Q^{0t}.

5. Estimating the elasticity of substitution

A number of methods have been proposed in the literature for estimating the elasticity of substitution from observable price and quantity data, assuming CES preferences. In this section we derive two methods, an algebraic method that uses matched-model Jevons and price and quantity indexes, and (two variants of) a regression method. We start with the algebraic method.[8]

The matched-model counterpart to the Jevons-type representation (11) of the CES index is

$$P_{MCES}^{0t} = \left[\frac{P_{MJ}^{0t} Q_{MJ}^{0t}}{V_{M(0t)}^{t} \big/ V_{M(0t)}^{0}} \right]^{\frac{1}{\sigma - 1}} P_{MJ}^{0t}, \tag{26}$$

where $P_{MJ}^{0t} = \prod_{i \in I_{M(0t)}} (p_i^t / p_i^0)^{1/N_{M(0t)}}$ and $Q_{MJ}^{0t} = \prod_{i \in I_{M(0t)}} (x_i^t / x_i^0)^{1/N_{M(0t)}}$ are the matched-model Jevons price and quantity indexes, and $N_{M(0t)}$ is the number of matched products between periods 0 and *t*. Equations (22) and (26) are both representations of the matched-model CES price index, and so we have

$$\left[\frac{P_{MJ}^{0t} Q_{MJ}^{0t}}{V_{M(0t)}^{t} \big/ V_{M(0t)}^{0}} \right]^{\frac{1}{\sigma - 1}} P_{MJ}^{0t} = \left[\frac{P_{MT}^{0t} Q_{MT}^{0t}}{V_{M(0t)}^{t} \big/ V_{M(0t)}^{0}} \right]^{\frac{1}{\sigma - 1}} P_{MT}^{0t}. \tag{27}$$

Equality (27) implies $[P_{MJ}^{0t} / P_{MT}^{0t}]^{\sigma} = Q_{MT}^{0t} / Q_{MJ}^{0t}$. Taking logarithms of both sides and some rearranging retrieves the elasticity of substitution:[9]

$$\sigma = -\frac{\ln\left(Q_{MJ}^{0t} / Q_{MT}^{0t}\right)}{\ln\left(P_{MJ}^{0t} / P_{MT}^{0t}\right)} \quad (P_{MJ}^{0t} \neq P_{MT}^{0t}). \tag{28}$$

Using all the "direct" price and quantity indexes going from the base period 0 to period t $(t = 1,...,T)$, equation (28) yields T estimates of σ. While there is a single "true" value for σ, due to disturbances the estimates are likely to differ, perhaps substantially. The average of the T estimates could be used to calculate the adjusted CES price index

[8] Opperdoes (2001) and De Haan (2002; 2005) implemented one of the methods proposed by Balk (1999). This method has no analytical solution, and the value of the elasticity of substitution must be approximated numerically.

[9] Instead of the bilateral (matched-model) Jevons and Törnqvist indexes we could also use the multilateral GEKS-Jevons and GEKS-Törnqvist indexes. The corresponding expression for the elasticity of substitution is $\sigma = -\ln\left(Q_{GEKSMJ}^{0t} / Q_{GEKSMT}^{0t}\right) / \ln\left(P_{GEKSMJ}^{0t} / P_{GEKSMT}^{0t}\right)$.

according to equation (24). Note that more estimates of σ could be found by using all bilateral comparisons across the sample period (the elements in GEKS). However, if the window length is large enough, the T estimates will probably suffice. Notice further that highly volatile or implausible, such as negative or extremely high, estimated values of σ point to problems with the validity of the assumptions when applying CES to real-world data.

A potential problem with the above algebraic method is that if $P_{MJ}^{0t} \simeq P_{MT}^{0t}$ for some period t, then the denominator of (28) will be close to zero, which could lead to an extreme estimate of σ and an adverse effect on the average estimated value. Also, the algebraic method does not allow for random disturbances. Ivancic, Diewert and Fox (2010) and De Haan (2019) discussed regression methods for estimating σ in order to deal with random disturbances. We propose two versions of an alternative regression method.

Our starting point is the matched-model version of equation (5), i.e.,

$$\left(\frac{s_{iM(0t)}^{t}}{s_{iM(0t)}^{0}} \right)^{\frac{1}{\sigma-1}} \frac{p_i^t}{p_i^0} = P_{MCES}^{0t}, \tag{29}$$

where P_{MCES}^{0t} is the matched-model CES index given by (20); the (normalized) matched-model expenditure shares are equal to $s_{iM(0t)}^{0} = p_i^0 x_i^0 / \sum_{i \in I_{M(0t)}} p_i^0 x_i^0 = s_i^0 / \sum_{i \in I_{M(0t)}} s_i^0$ and $s_{iM(0t)}^{t} = p_i^t x_i^t / \sum_{i \in I_{M(0t)}} p_i^t x_i^t = s_i^t / \sum_{i \in I_{M(0t)}} s_i^t$. Taking logarithms of both sides of (29) and rearranging terms we find

$$\ln\left(\frac{s_{iM(0t)}^{t}}{s_{iM(0t)}^{0}} \right) = (\sigma - 1)\ln P_{MCES}^{0t} + (1-\sigma)\ln\left(\frac{p_i^t}{p_i^0} \right). \tag{30}$$

Setting $1-\sigma = \beta$ and $(\sigma-1)\ln P_{MCES}^{0t} = \alpha^t$, and adding an error term ε_i^t with zero mean, we obtain the following regression model, which must be estimated on the data of each period t $(t = 1,...,T)$ separately:

$$\ln\left(\frac{s_{iM(0t)}^{t}}{s_{iM(0t)}^{0}} \right) = \alpha^t + \beta \ln\left(\frac{p_i^t}{p_i^0} \right) + \varepsilon_i^t. \tag{31}$$

We denote the Ordinary Least Squares (OLS) parameter estimates by $\hat{\alpha}^t$ and $\hat{\beta}_{M(0t)}$. The estimated value of σ from the period t (and period 0) data is equal to $\hat{\sigma}_{M(0t)} = 1 - \hat{\beta}_{M(0t)}$. Like with the algebraic method we obtain T different estimates of σ, and it makes sense to use the average value to calculate the adjusted CES index given by equation (24). There is a caveat, however. The number of observations for each regression, i.e., the number of matched products $N_{M(0t)}$, tends to decline over time so that that the parameter estimates, hence the estimates of σ, may become less reliable for increasing *t*.

Interestingly, the regression coefficients also provide us with an estimate of the period *t* matched-model CES price index: $\hat{P}^{0t}_{MCES} = 1/\exp(\hat{\alpha}^t / \hat{\beta}_{M(0t)})$. Using the property that the regression residuals sum to zero in every period, it is easy to show that $\hat{P}^{0t}_{MCES}$ can be written as

$$\hat{P}^{0t}_{MCES} = \left[\frac{\prod_{i \in I_{M(0t)}} (s_i^t)^{\frac{1}{N_{M(0t)}}}}{\prod_{i \in I_{M(0t)}} (s_i^0)^{\frac{1}{N_{M(0t)}}}} \right]^{\frac{1}{\hat{\sigma}_{M(0t)}-1}} P^{0t}_{MJ} = \left[\frac{P^{0t}_{MJ} Q^{0t}_{MJ}}{V^t_{M(0t)} / V^0_{M(0t)}} \right]^{\frac{1}{\hat{\sigma}_{M(0t)}-1}} P^{0t}_{MJ} . \tag{32}$$

The right-hand side of (32) is equal to the Jevons-type representation (26) of the matched-model CES price index (Redding and Weinstein's (2020) CES Common Varieties Index), evaluated at $\hat{\sigma}_{M(0t)}$ (which varies across time).

The use of OLS regression to estimate model (31) seems appropriate if the errors have constant variance. If the errors are heteroskedastic, or if we do not want to give equal weight to all the products for another reason, the model can be estimated using Weighted Least squares (WLS) regression. Let us denote the WLS parameter estimates by $\tilde{\alpha}^t$ and $\tilde{\beta}_{M(0t)}$. The estimated value of σ from the period *t* (and period 0) data is of course equal to $\tilde{\sigma}_{M(0t)} = 1 - \tilde{\beta}_{M(0t)}$. Again we obtain *T* different estimates of σ.

The period *t* CES price index that corresponds with the WLS parameter estimates is $\tilde{P}^{0t}_{MCES} = 1/\exp\left(\tilde{\alpha}^t / \tilde{\beta}_{M(0t)}\right)$. Using WLS regression, the weighted residuals sum to zero. Suppose we would use the average of the matched expenditure shares in the two periods compared, $(s^0_{iM(0t)} + s^t_{iM(0t)})/2$, as regression weights. It can be shown that $\tilde{P}^{0t}_{MCES}$ can then be written as

$$\tilde{P}^{0t}_{MCES} = \left[\frac{\prod_{i \in I_{M(0t)}} (s^t_{iM(0t)})^{\frac{s^0_{iM(0t)} + s^t_{iM(0t)}}{2}}}{\prod_{i \in I_{M(0t)}} (s^0_{iM(0t)})^{\frac{s^0_{iM(0t)} + s^t_{iM(0t)}}{2}}} \right]^{\frac{1}{\tilde{\sigma}_{M(0t)}-1}} P^{0t}_{MT} = \left[\frac{P^{0t}_{MT} Q^{0t}_{MT}}{V^t_{M(0t)} / V^0_{M(0t)}} \right]^{\frac{1}{\tilde{\sigma}_{M(0t)}-1}} P^{0t}_{MT} . \tag{33}$$

The right-hand side of (33) equals the Törnqvist-type representation (22) of the matched-model CES index, evaluated at $\tilde{\sigma}_{M(0t)}$.

The elasticity of substitution is unlikely to be constant across all products; some products are highly substitutable but others are not. In most empirical work, separate CES indexes for different product categories are estimated, with each product category having its own elasticity of substitution.[10] This will also be our approach in the empirical Section 6 below.

6. Empirical illustration

6.1 The data set

We utilize the IRI Marketing Data Set which was made available for academic research (Bronnenberg, Kruger and Mela, 2008). Our full data set contains six years of weekly scanner data, from 2001 to 2007, for 31 product categories sold by chain grocery and drug stores in 47 U.S. submarkets.[11] A CPI is typically compiled on a monthly basis (although not in New Zealand where the CPI is still compiled quarterly). Weeks in the data set are identified by the date in the data set which relates to a fixed day of the week. Using those dates we assigned weeks to calendar months. This means that (what we call) a month can actually exist of four of five weeks of data.

To be able to include strongly seasonal goods, the window for calculating GEKS and adjusted GEKS indexes should be at least 13 months long. Most statistical agencies work with a window length of 25 months, and we will do this too. We wanted to use the most recent data, but due to what seemed to be a product identification issue in 2007 (and which led to a significant lack of matching) we decided to use data from December 2004 to December 2006.

Panel A.1 in Appendix 2 shows the number of products sold per month for each of the 31 product categories. Individual products are identified by combining barcode, or actually the associated UPC, Universal Product Code, the U.S. version of GTIN, Global

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[10] Balk (1999) argued that in a two-level CES framework with Feenstra adjustments for product churn, the lower-level elasticities of substitution must be greater than 1 and the upper-level elasticity smaller than 1, which is consistent with most empirical evidence. Note that we will not estimate the upper-level elasticity of substitution.

[11] Statistics New Zealand purchased the data at the time for price measurement research. The IRI data set was later extended to 2012 (Kruger and Pagni, 2015) but Statistics New Zealand did not purchase the new data.

Trade Item Number) or SKU (Stock Keeping Unit) and vendor id. This means we treat the same UPC or SKU sold by different vendors as different goods. That is, we aggregate across submarkets so that product level expenditure, quantity (in terms of units sold) and unit value price (expenditure divided by quantity) relate to a UPC/SKU in a particular vendor. As can be seen from Panel A.1, the total number of products per category is quite stable over time. Note that the full names of the product categories can be found in Table 1 below.

Of course, the total number of products per category does not tell us much about product churn. Panel A.2 in Appendix 2 provides some information on churn – it shows the number of matched products per category between the fixed base period of December 2004 and the 24 comparison months. For many product categories the degree of churn is substantial and for some it is rather extreme: for example, for "toilet tissue" only 40% of the products that were sold in December 2004 were still sold two years later. Panel A.2 also shows the corresponding aggregate expenditure shares of the matched product in the comparison months. For most product categories the decrease in the matched expenditure shares is bigger than the decline in the total number of matched products.

The way in which we define individual products affects measured product churn. For example, it has been argued that identifying products by UPC/GTIN is likely to lead to over-estimation of the "true" rate of churn. The main reason is that a UPC may change even if the product stays the same from the consumers' perspective, for example in case of a slight change in packaging. We will come back to this topic in the discussion Section 7 below.

6.2 Results

As mentioned above, the set of matched products often becomes smaller for more distant months, and the estimates of the elasticity of substitution are likely to become less reliable since they rely on fewer observations. The use of an unweighted average of the estimated values for all months in the calculation of the Feenstra adjustment term may therefore not be optimal. Also, it could be useful to delete outliers or apply some smoothing procedure. However, to keep things simple in this empirical illustration, we decided to use the 24-month unweighted arithmetic average value of the estimated elasticity of substitution for all of the product categories. Table 1 shows the results from the OLS and WLS regression methods and the algebraic (ALG) method explained in Section 5.

Table 1. Average estimated elasticity of substitution

	OLS	WLS	ALG
Beer	1.43	3.51	10.44
Blades	0.15	0.03	**-10.23**
Carbonated beverages	2.04	2.90	5.16
Cigarettes	**-0.25**	0.60	**-5.67**
Coffee	1.10	1.79	**-0.80**
Cold cereal	2.87	3.73	2.46
Deodorant	0.40	1.01	**-54.66**
Diapers	0.08	**-1.03**	**-28.87**
Facial tissues	1.89	0.59	15.95
Frozen dinners/entrees	2.62	3.43	287.12
Frozen pizza	1.80	2.78	13.39
Household cleaner	1.43	3.51	10.44
Hotdogs	2.47	1.35	**-1.93**
Laundry detergent	2.70	4.26	30.02
Margarine/butter blends	1.88	2.37	**-16.09**
Mayonnaise	1.66	4.07	0.30
Milk	1.05	0.85	0.22
Mustard and ketchup	0.58	3.07	3.55
Paper towels	2.55	3.87	16.04
Peanut butter	0.35	2.24	1.56
Photo supplies	1.43	3.51	10.44
Razors	**-0.44**	**-7.15**	36.06
Salty snacks	2.15	10.88	5.48
Shampoo	0.11	**-0.11**	**-26.22**
Soup	1.49	2.04	6.91
Spaghetti/Italian sauce	1.76	2.58	2.21
Sugar substitutes	**-0.14**	0.72	5.45
Toilet tissue	2.06	8.26	**-65.71**
Toothbrushes	0.18	1.00	**-23.55**
Toothpaste	1.51	2.81	39.72
Yogurt	2.18	2.50	**-0.21**

In most cases the average WLS values in Table 1 are higher than the average OLS values. The OLS values are often rather low; in general the WLS values look a bit more plausible. Negative values (given in bold), which are inconsistent with theory, are found in some instances. The algebraic estimates range from -65.71 for "toilet tissue" to +287.12 for "frozen dinners/entrees". Most of the algebraic estimates are implausible, really – the algebraic approach to estimating the elasticity of substitution does not seem to perform well.

Panel A.3 in Appendix 2 contains the 24 estimates for all product categories from which the average values are calculated. Volatility is high for many product categories. The estimates based on the algebraic method in particular can be very volatile. The most extreme example is "carbonated beverages" in Figure 1, where the algebraic values range from -60.02 to +128.85. The high volatility corroborates the unsuitability of the algebraic method. The regression methods perform better, but the estimates are still quite volatile. Figure 2 shows the results for "cold cereal". This figure also illustrates the fact that the OLS estimates are usually lower than the WLS estimates for which we do not yet have a good explanation.

Figure 1: Elasticity of substitution, Carbonated beverages

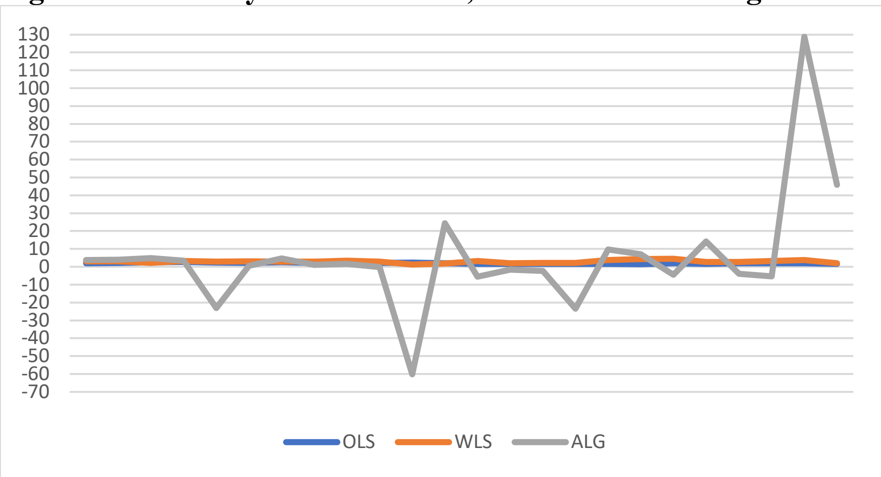
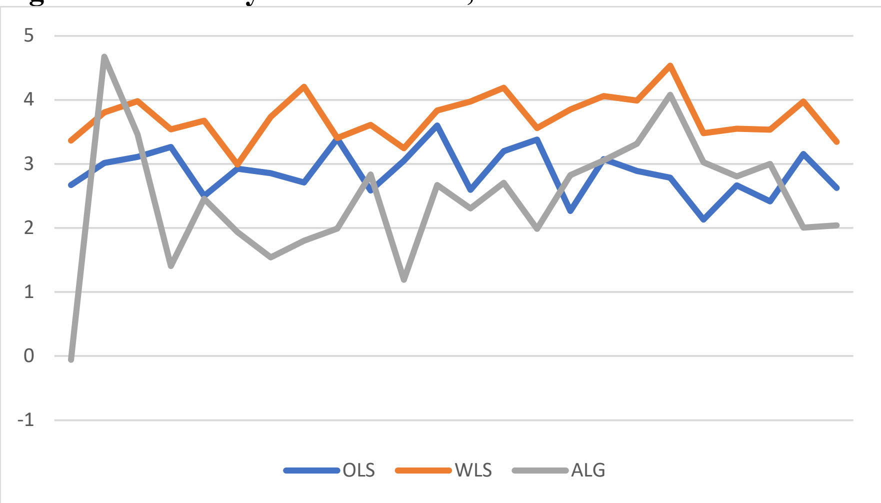


Figure 2: Elasticity of substitution, Cold cereal



Panel A.4 in Appendix 2 contains matched-model GEKS-Törnqvist (and GEKS-Jevons) price indexes and the Feenstra adjusted GEKS-Törnqvist price indexes based on the average elasticity estimates from the algebraic method and the regression methods for all product categories. Some of the adjusted indexes are really implausible. This behavior is driven by the combination of a low degree of matching and a low average value of the elasticity of substitution. Note that the adjusted GEKS-Törnqvist index based on the WLS method for estimating the elasticity is almost zero for "deodorant" and almost infinite for "toothbrushes in most periods due to the fact that the average estimate of the elasticity is very close to 1. Notice that for most categories the difference between the matched-model GEKS-Törnqvist price index and the matched-model GEKS-Jevons price index is much smaller than the difference between the matched-model GEKS-Törnqvist price index and the adjusted versions.

Figures 3 and 4 copy the adjusted GEKS-Törnqvist price indexes for "toilet tissue" and "frozen dinners/entrees" based on the average elasticity estimates from the algebraic method and the regression methods. The standard matched-model GEKS-Törnqvist price indexes are plotted too. A property of the Feenstra method is that extreme values of the elasticity of substitution tend to lead to very small adjustment terms. This causes the small adjustments for the algebraic method with respect to the standard GEKS-Törnqvist price index for "toilet tissue" and "frozen dinners/entrees".

The average OLS and WLS estimates of the elasticity of substitution for "toilet tissue" (2.06 and 8.26) and "frozen dinners/entrees" (2.62 and 3.43) are obviously more realistic than the extreme algebraic values. As a result, the (downward) adjustment terms based on the regression approaches to estimating the elasticity in Figures 3 and 4 are much bigger. In fact the OLS-based adjusted GEKS-Törnqvist index for "toilet tissue" does not make any sense. This result is driven by a severe lack of matching in combination with a relatively low value of the elasticity of substitution (2.06).

Figure 3: Standard and adjusted GEKS-Törnqvist price indexes, Toilet tissue

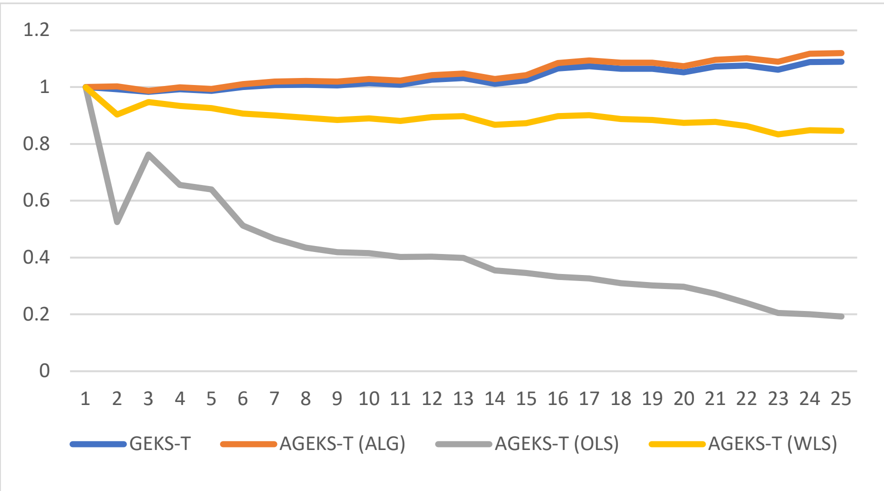
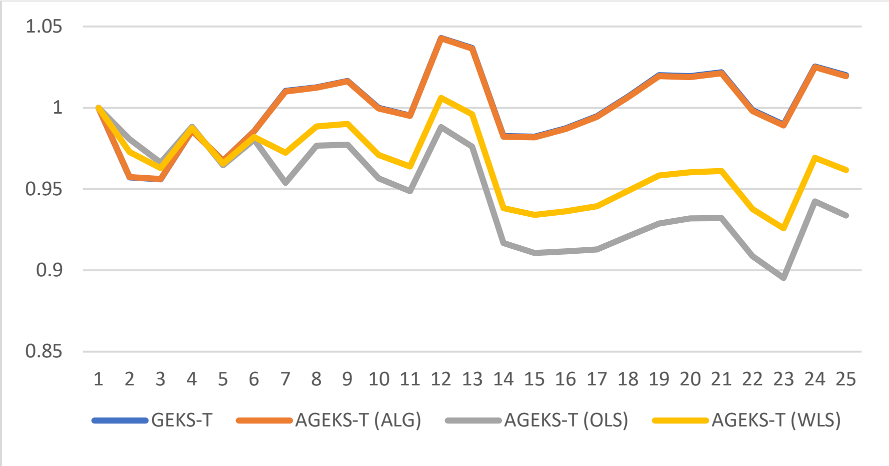


Figure 4: Standard and adjusted GEKS-Törnqvist price indexes, Frozen dinners/entrees



Figures 5 and 6 copy the standard matched-model and adjusted GEKS-Törnqvist price indexes for two other product categories, “carbonated beverages” and “cold cereal”. Here, the average estimates of the elasticity of substitution from the regression methods and the algebraic method all seem plausible; the OLS, WLS and algebraic-based values in Table 1 were 2.04, 2.90 and 5.16 for “carbonated beverages” and 2.87, 3.73, and 2.46 for “cold cereal”. Figures 5 and 6 nicely show that a higher value of the elasticity brings the adjusted index closer to the unadjusted index (given the matched expenditure shares). Again, this is a property of the Feenstra method. The plausibility of the adjustments is not easy to assess, but the magnitude of the difference between the adjusted indexes for values of the elasticity that do not differ much is somewhat worrying.

Figure 5: Standard and adjusted GEKS-Törnqvist price indexes, Carbonated beverages

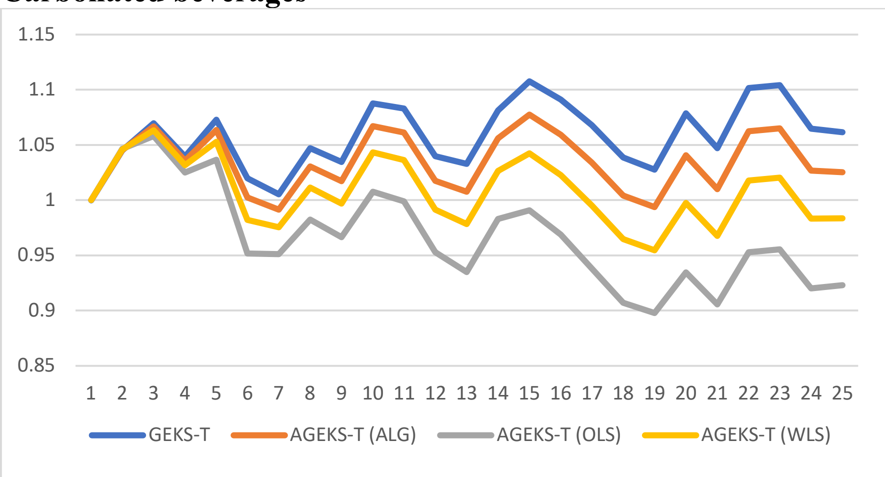
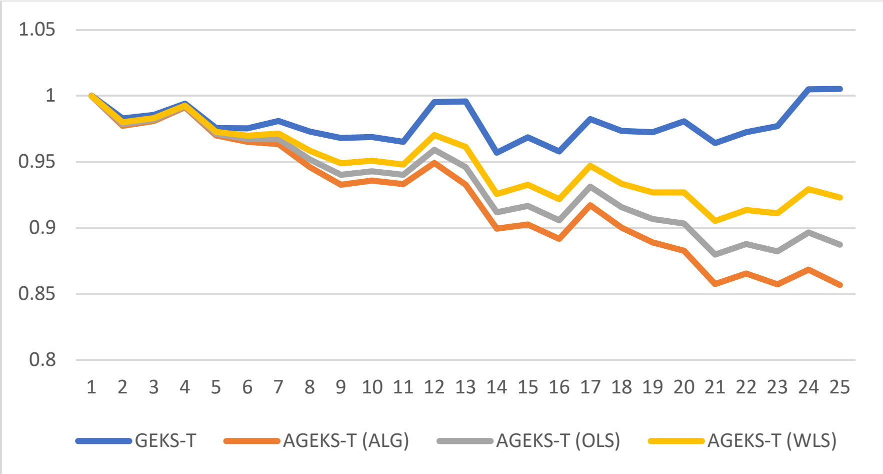


Figure 6: Standard and adjusted GEKS-Törnqvist price indexes, Cold cereal



7. Discussion

Our adjusted GEKS-Törnqvist method produces a transitive price index that accounts for substitution effects, because the building blocks are superlative Törnqvist price indexes, and for new and disappearing product varieties via the Feenstra (1994) adjustment term. When applying the method to real-world (IRI) scanner data we found some implausible results. There are a number of conceptual and practical issues that may have contributed to these findings.

We start by mentioning that Feenstra (1994) actually proposed period-on-period chaining; at the time, chain drift was not seen as a big problem. An advantage of chaining, in addition to the high degree of matching between adjacent periods, is the treatment of changes in the taste parameters. Taste changes should not affect measured price change, and so the convention is to hold the taste parameters fixed in bilateral price comparisons. In the GEKS context, the taste parameters are kept fixed in all the bilateral comparisons across the entire sample period $(0,\ldots,T)$.

Like other multilateral indexes, GEKS indexes are revisable: when data is added for period T+1, the newly estimated numbers for periods $1,\ldots,T$ generally differ from the previously estimated numbers. To avoid such revisions, statistical agencies use a rolling-window approach combined with a form of splicing.[12] The same approach can be used

[12] For an extensive study using scanner data on the impact of various splicing methods for GEKS-Törnqvist and other multilateral price indexes, see Fox, Levell and O'Connell (2023).

for the adjusted GEKS-Törnqvist index. Splicing raises some issues because transitivity is no longer satisfied and splice drift can arise. Splicing does mean that taste changes will gradually be accounted for.

The Feenstra adjustment is a simple way of accounting for new and disappearing products, assuming CES preferences. There are other ways, in particular via imputation of the "missing prices". In the Introduction we already mentioned the possibility of using hedonic regressions. Since CES is a demand-oriented approach, it can be argued that we should try to estimate the (Hicksian) demand reservation prices, i.e., the prices that would drive demand to zero. Diewert and Feenstra (2021) have shown that the reservation prices which are consistent with the Feenstra adjustment are infinitely high. This is unrealistic; demand will already be driven down to zero when prices are far below infinity.

The idea of a constant elasticity of substitution can also be criticized. Differences in quality will remain within a product category that consists of broadly similar products in terms of price-determining characteristics. So, a certain product will be more similar to some products than to others and disaggregation according to product similarity likely leads to higher substitution possibilities and higher values of the elasticity of substitution within sub-categories; see also Diewert (1974) and Ehrlich et al. (2020). The assumption of the elasticity of substitution being the same for all pairs of products (and constant over time) is probably too restrictive, even for broadly similar products.

Our main theoretical result is the adjusted GEKS-CES index given by (24). Using the results in Sections 5 and 6, and again assuming that the CES assumptions hold exactly true, it is easy to verify that an alternative version of the adjusted GEKS-CES index is

$$P_{GEKSCES(A)}^{0t} = \left[\frac{P_{GEKSMJ}^{0t} Q_{GEKSMJ}^{0t}}{V^t / V^0} \right]^{\frac{1}{\sigma - 1}} P_{GEKSMJ}^{0t}, \quad (34)$$

where P_{GEKSMJ}^{0t} and Q_{GEKSMT}^{0t} are GEKS-Jevons price and quantity indexes with matched-model bilateral Jevons indexes as elements. In the future we want to estimate this index too and compare it with (24).[13]

[13] Representation (34) of the adjusted GEKS-CES index can be viewed as the multilateral version of the Redding and Weinstein's (2020) CUPI (CES Unified Price Index) while holding the taste parameters fixed. The CUPI allows the taste parameters to change over time. Abe and Rao (2022) argued that non-fixity of the taste parameters makes the CUPI transitive, in which case the use of GEKS would be superfluous. We do not agree though with the view that measured inflation should reflect not only changes in prices but also changes in taste.

The proposed adjusted GEKS-CES index requires that the observable expenditure shares are equal to the "optimal" CES expenditure shares, or at least that any disturbances are random. This will not be the case in particular circumstances, however, for instance if consumers stockpile goods that are on sale. Thus, although our price index is transitive and does not suffer from chain drift due to sales, we know that an important assumption may be violated.

As mentioned earlier, the CES price index is not defined for $\sigma = 1$, but we know that for $\sigma \to 1$ the matched-model CES index equals the matched-model Törnqvist price index. Similarly, for $\sigma \to 1$ the matched-model GEKS-CES index should be equal to the matched-model GEKS-Törnqvist price index. The Feenstra adjustment is not defined for $\sigma = 1$ either. The behavior of the adjusted GEKS-Törnqvist price index when σ is close to 1 seems to be unpredictable. This is also because the Feenstra adjustment term depends on the matched expenditure shares in the periods compared and the value of the elasticity of substitution, both of which suffer from practical measurement problems.

One practical/conceptual problem we alluded to earlier is how individual products are defined. The choice between, e.g., UPC and SKU can make a significant difference to the measurement of matched expenditure shares and hence to the Feenstra adjustment term. This problem becomes especially important if the elasticity of substitution is close to 1. However, statistical agencies often do not have a choice and just have to work with the available identifier. In our case, products have been defined by a mixture of UPC and SKU. Note that if the identifier (such as UPC) is "too detailed" in the sense that it can change while the product remains essentially the same from the consumers' perspective, and if price increases happen when during such unimportant changes, then any matched-model price index, including the matched-model Törnqvist, will suffer from downward bias.

Even if we do not worry too much about conceptual issues and the identifier issue, the estimation of the elasticity of substitution is really problematic. Different estimation methods led to quite different and often totally implausible results. The estimates were very volatile too; similar findings have been reported by others, e.g., Ivancic, Diewert and Fox (2010). Uncertainty about the value of the elasticity of substitution is likely to hamper the use of our method in official price measurement. Statistical agencies should be aware though that not adjusting for product churn can introduce new/disappearing goods bias, or quality-change bias, in the CPI.

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Appendix 1: Examples of representations of the CES index

As shown by equation (6) in the main text, there are an infinite number of representations of the CES price index under the (strict) assumptions that the prices are measured without error and the expenditure shares exactly satisfy (3). Using a fixed set of products, I, this Appendix presents examples for three different values of σ: 0, 2 and 3. We believe that the expressions for the values 2 and 3 are novel. For convenience, we repeat equation (6) but exclude the geometric form:

$$P_{CES}^{0t} = \sum_{i \in I} a_i^t \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{\sigma - 1}} \left(\frac{p_i^t}{p_i^0} \right) = \left[\sum_{i \in I} a_i^t \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{1-\sigma}} \left(\frac{p_i^t}{p_i^0} \right)^{-1} \right]^{-1}, \tag{A.1}$$

with weights summing to unity; $\sum_{i \in I} a_i^t = 1$.

For $\sigma = 0$ and setting $a_i^t = s_i^t$ in the first expression of equation (A.1), we obtain the Laspeyres price index

$$P_{CES}^{0t} = \sum_{i \in I} s_i^t \left(\frac{s_i^0}{s_i^t} \right) \left(\frac{p_i^t}{p_i^0} \right) = \sum_{i \in I} s_i^0 \left(\frac{p_i^t}{p_i^0} \right) = \frac{\sum_{i \in I} p_i^t x_i^0}{\sum_{i \in I} p_i^0 x_i^0} = P_L^{0t} . \quad \text{(A.2)}$$

Setting $a_i^t = s_i^0$ in the second expression of (A.1) yields the Paasche price index:

$$P_{CES}^{0t} = \left[\sum_{i \in I} s_i^0 \left(\frac{s_i^0}{s_i^t} \frac{p_i^t}{p_i^0} \right)^{-1} \right]^{-1} = \left[\sum_{i \in I} s_i^t \left(\frac{p_i^t}{p_i^0} \right)^{-1} \right]^{-1} = \frac{\sum_{i \in I} p_i^t x_i^t}{\sum_{i \in I} p_i^0 x_i^t} = P_P^{0t} . \quad \text{(A.3)}$$

These results are well known but usually derived in a different way.

Other choices as weights are also possible. For instance, we could use the period t quantity shares $a_i^t = x_i^t / \sum_{i \in I} x_i^t$ in the first expression of (A.1), even though adding up quantities of heterogeneous products is not economically meaningful. This gives

$$P_{CES}^{0t} = \sum_{i \in I} \frac{x_i^t}{\sum_{i \in I} x_i^t} \left(\frac{s_i^0}{s_i^t} \right) \left(\frac{p_i^t}{p_i^0} \right) = \sum_{i \in I} \frac{x_i^t}{\sum_{i \in I} x_i^t} \left(\frac{x_i^0 \big/ \sum_{i \in I} p_i^0 x_i^0}{x_i^t \big/ \sum_{i \in I} p_i^t x_i^t} \right) = \frac{\sum_{i \in I} p_i^t x_i^t \big/ \sum_{i \in I} x_i^t}{\sum_{i \in I} p_i^0 x_i^0 \big/ \sum_{i \in I} x_i^0} = P_{UV}^{0t} . \quad \text{(A.4)}$$

Thus, for $\sigma = 0$ the CES index can be expressed as Laspeyres and Paasche price indexes, and hence as the Fisher ideal price index, but (surprisingly perhaps) also as the unit value index, P_{UV}^{0t}.

For $\sigma = 2$ and using the period 0 expenditure shares as weights, $a_i^t = s_i^0$, in the first expression of (A.1) yields

$$P_{CES}^{0t} = \sum_{i \in I} s_i^0 \left(\frac{s_i^t}{s_i^0} \right) \left(\frac{p_i^t}{p_i^0} \right) = \sum_{i \in I} s_i^t \left(\frac{p_i^t}{p_i^0} \right) = P_{Pal}^{0t} , \quad \text{(A.5)}$$

which is known as the Palgrave price index, P_{Pal}^{0t}.

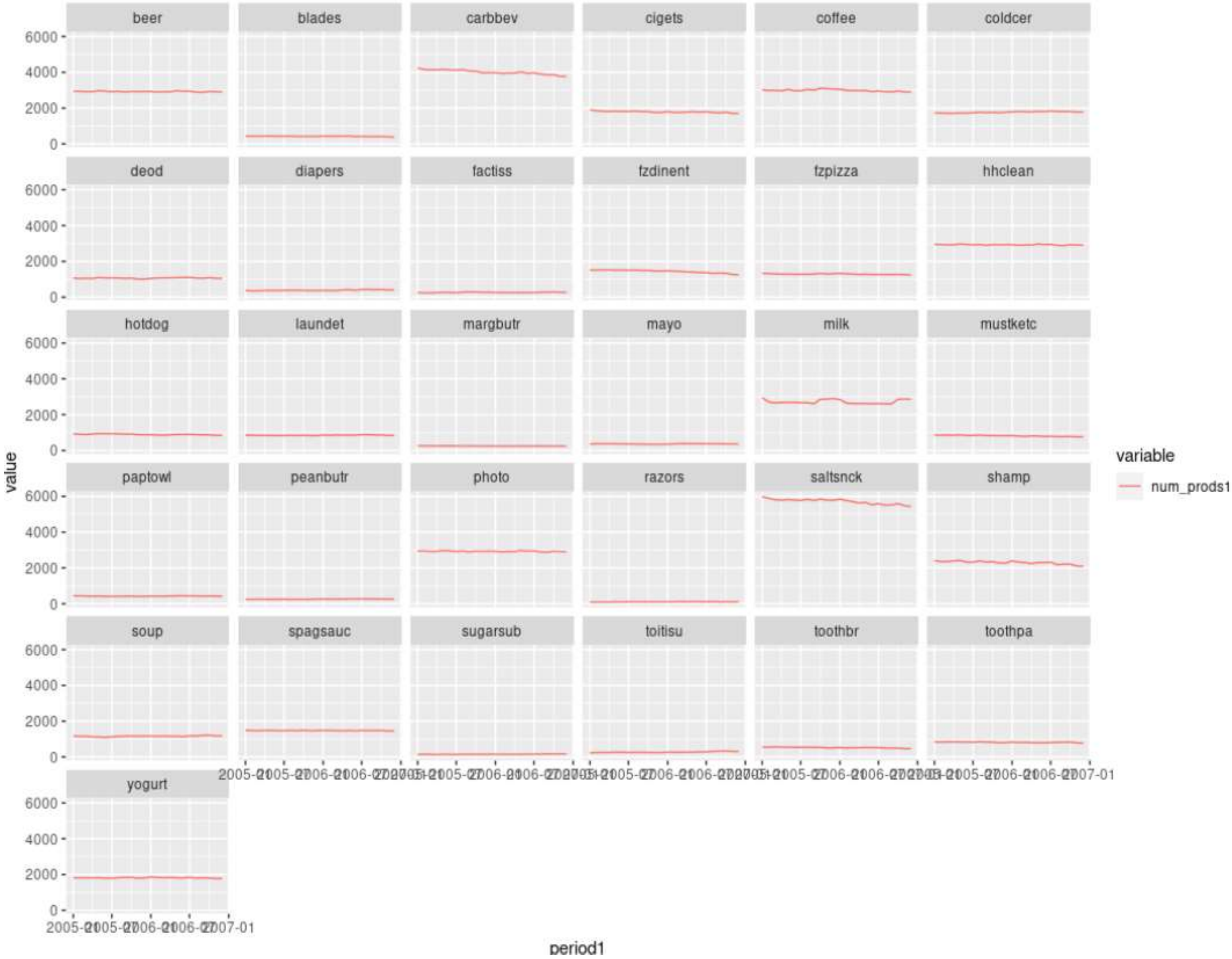
For $\sigma = 3$ and again using $a_i^t = s_i^0$ in the first expression of (A.1), we find

$$P_{CES}^{0t} = \sum_{i \in I} s_i^0 \left(\frac{s_i^t}{s_i^0} \right)^{\frac{1}{2}} \left(\frac{p_i^t}{p_i^0} \right) = \sum_{i \in I} (s_i^0 s_i^t)^{1/2} \left(\frac{p_i^t}{p_i^0} \right). \quad \text{(A.6)}$$

This expression does not have a name in the standard index number literature because it is not a proper price index: the geometric averages of the expenditure shares in periods 0 and t do not add up to 1 so that the index violates the proportionality (in prices) test if the CES assumptions are not satisfied.

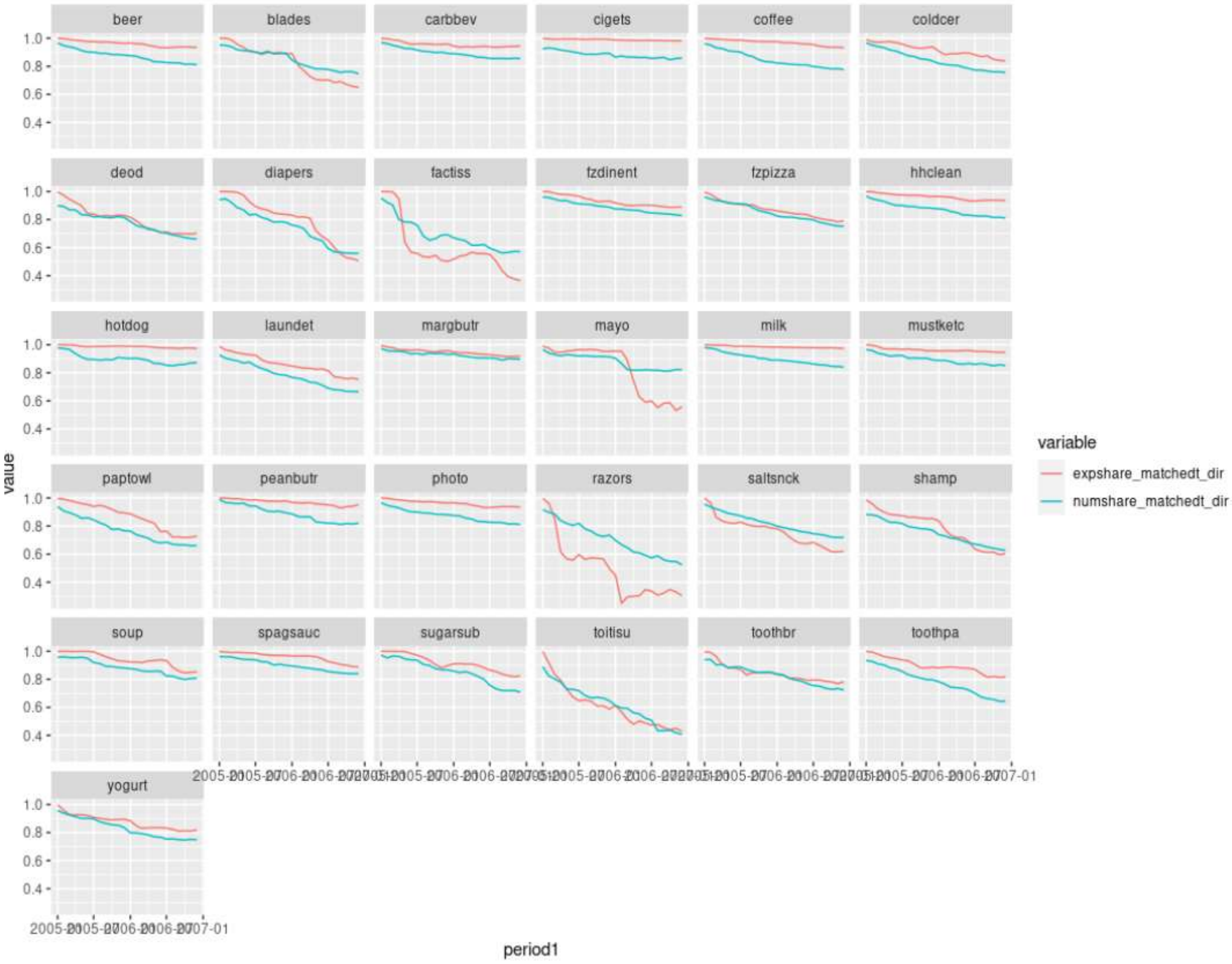
Appendix 2: Data and results for all product categories

Panel A.1: Number of products



Explanation: number of individual products sold, as identified by UPC (Universal Product Code, barcode).

Panel A.2: Number of matched products and matched expenditure shares

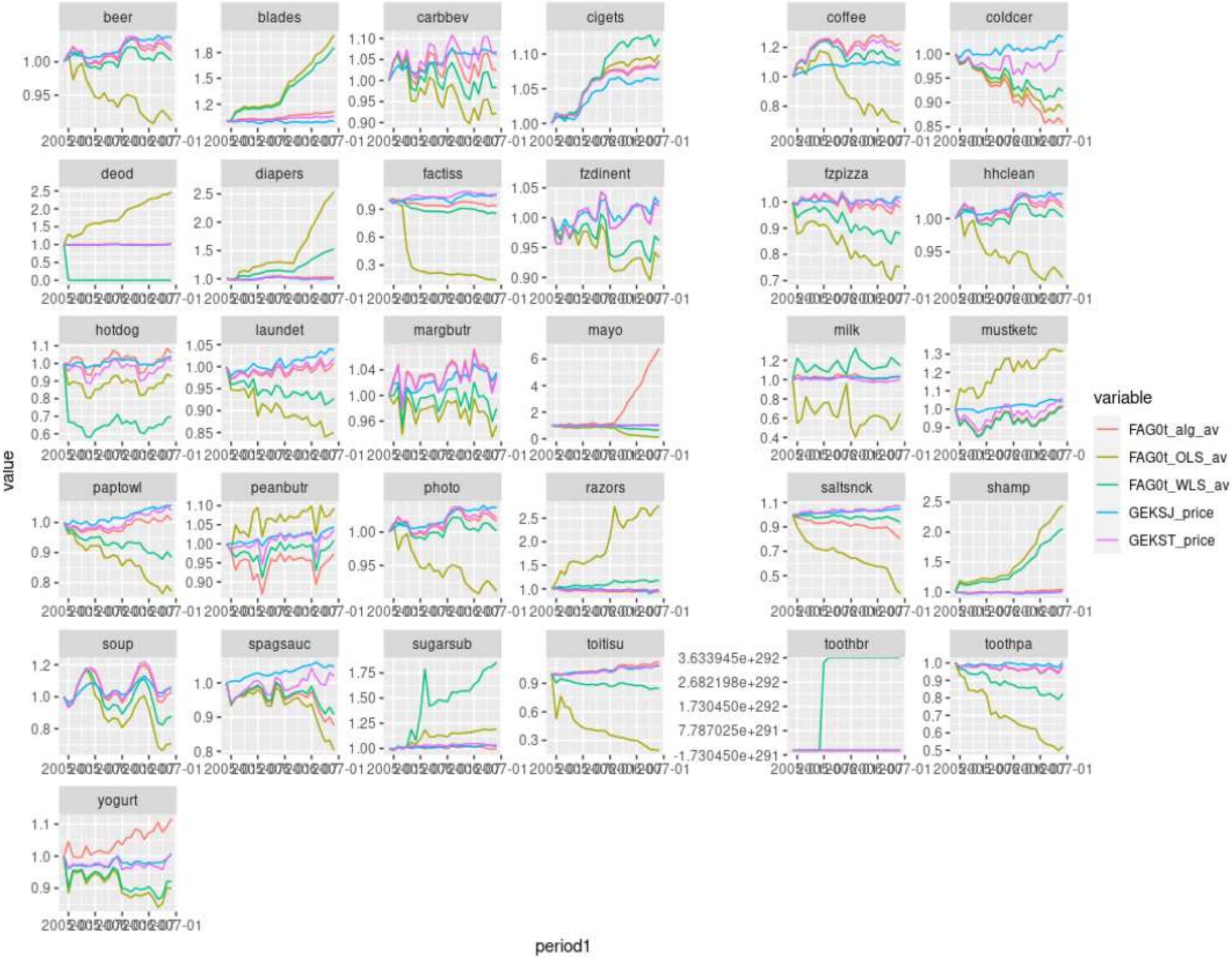


Explanation: number of matched products between all of the comparison months, January 2005 to December 2006, and the (fixed) base month of December 2004 and the corresponding matched expenditure shares in the comparison months.

Panel A.3: Estimated elasticities of substitution

Explanation: values of the elasticity of substitution according to three estimation methods (OLS and WLS regression methods and algebraic method).

Panel A.4: Matched-model and adjusted price indexes



Explanation: matched-model GEKS-Törnqvist and GEKS-Jevons price indexes; adjusted GEKS-Törnqvist based on the three approaches to estimating the elasticity of substitution (OLS, WLS and algebraic), December 2004 to December 2006.