Evolution of the GEKS index

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18^{th} Meeting of the Ottawa Group, Ottawa, 13 - 15 May, 2024

May 7, 2024

Abstract

The GEKS index is a well-known multilateral index, which is used by many statisticians in measuring inflation based on scanner data. As a rule, it is assumed that the underlying index, being in the body of the GEKS formula, satisfies the time reversal test. It is also most often assumed that the underlying index is superlative and therefore the GEKS index based on the Fisher, Törnqvist (GEKS-T or CCDI) or Walsh formula (GEKS-W) is most often considered. However, the 'classic' GEKS index does not meet the stringent identity test. Unfortunately, most of the known multilateral indices do not meet the time reversal test, which is seen by many researchers as their weakness.

This paper reviews more or less well-known modifications and generalizations of the GEKS index. We discuss three new and general classes of indices based on the idea of GEKS method and some special cases of these classes, which include GEKS, GEKS-T and GEKS-W indices and additionally GEKS-L and GEKS-GL indices. One class uses elasticity of substitution so is close to the economic approach, while another class - like the multilateral Geary-Khamis index - uses quality-adjusted prices and quantities. Not all of the GEKS modifications discussed require the underlying index to meet the time reversal test. It is shown that where this assumption has been dropped, an identity test has been gained. The basic axiomatic properties of the proposed indices are presented and, in an empirical study, we compare them on the basis of real scanner data sets.

JEL codes: C43, C63, C81, D12
Keywords: inflation measurement, Consumer Price Index (CPI), scanner data, multilateral indices, identity test, GEKS method

[*]This paper represents the subjective opinion of the author and does not represent position of Statistics Poland on the content presented.

1 Introduction

Among the multitude of challenges facing statistical offices that use scanner data in inflation measurement is the problem of choosing a proper price index formula (Australian Bureau of Statistics (2016), Consumer Price Index Manual: Concepts and Methods, 2020). In general, bilateral unweighted and weighted price indices, chain indices and multilateral indices can be taken into consideration here. Nevertheless, the choice of price index should lead to a reduction in the substitution bias and chain drift bias (Chessa et al. 2017). Note that in the literature, there is a set of criteria that can be taken into account when choosing a price index formula. In particular, we can meet the *axiomatic*, *economic* and *stochastic approaches* (Eurostat 2022), as well as the complementary *time-consuming approach*, and the *new stochastic approach* (Białek & Beręsewicz 2021).

As it is commonly known, GEKS-type indices (e.g. the GEKS, CCDI, GEKS-W indices) have good axiomatic properties, are close to the *economic approach* since they are based on superlative indices. It can be shown that the GEKS index (Gini 1931, Eltetö & Köves 1964) is exact for a "flexible" functional form, i.e. it expresses the price differences experienced by optimising consumers without imposing restrictive assumptions about how they can substitute between products. Moreover, the GEKS-type indices are relatively fast in computations (Białek 2022*a*) which can be of practical importance when dealing with large scanner data sets. This paper reviews more or less well-known modifications and generalizations of the GEKS index. We discuss several general classes of indices based on the idea of GEKS method and some special cases of these classes, which include GEKS, GEKS-T and GEKS-W indices and additionally GEKS-L and GEKS-GL indices. One class uses elasticity of substitution so is close to the economic approach, while another class - like the multilateral Geary-Khamis index - uses quality-adjusted prices and quantities. Not all of the GEKS modifications discussed require the underlying index to meet the time reversal test. It is shown that where this assumption has been dropped, an identity test has been gained. The basic axiomatic properties of the proposed indices are presented and, in an empirical study, we compare them on the basis of real scanner data sets.

2 The GEKS-type index idea

Let us denote sets of homogeneous products belonging to the same product group in the months 0 and t by G_0 and G_t respectively, and let $G_{0,t}$ denote a set of matched products in both moments 0 and t. Let p_i^τ and q_i^τ denote the price and quantity of the i-th product at the time τ and let $N_{0,t}$ be the number of elements of set $G_{0,t}$.

The GEKS price index between the months 0 (the base period) and t (the current period) is an unweighted geometric mean of $T+1$ ratios of bilateral price indices $P^{\tau,t}$ and $P^{\tau,0}$, which are based on the same price index formula. Typically, the GEKS method uses the superlative Fisher (1922) price index, resulting in the following formula:

$$P_{GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_F^{\tau,t}}{P_F^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{1}$$

where $P_F^{\tau,t}$ denotes the Fisher price index calculated for products from the set $G_{\tau,t}$.

The GEKS method for making international index number comparisons comes from Gini (1931) but it was derived in a different manner by Eltetö & Köves (1964) and Szulc (1964). Feenstra et al. (2009), and also de Haan & van der Grient (2011) suggested that the Törnqvist (1936) price index formula could be used instead of the Fisher price index in the Gini methodology. Caves et al. (1982) used the GEKS idea with the Törnqvist index as a base in the context of making quantity comparisons across production units (the CCD method) and Inklaar & Diewert (2016) extended the CCD methodology to making price comparisons across production units. Consequently, in the article by Diewert & Fox (2018), the multilateral price comparison method involving the GEKS method based on the Törnqvist price index is called the CCDI method. The corresponding CCDI price index can be expressed as follows:

$$P_{CCDI}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_T^{\tau,t}}{P_T^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{2}$$

where $P_T^{\tau,t}$ denotes the Törnqvist price index calculated for products from the set $G_{\tau,t}$.

In the paper by Chessa et al. (2017), we can find a hint for selecting a base index formula for the GEKS method: "the bilateral indices should satisfy the time reversal test" but it is most often assumed that the price index formula found in the body of the GEKS index is a superlative formula (van Loon & Roels 2018, Diewert & Fox 2018). For this reason, a GEKS index based on the superlative Walsh (1901) index, which we will refer to as GEKS-W in the remainder of this paper, is also often considered. It can be written as follows:

$$P_{GEKS-W}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_W^{\tau,t}}{P_W^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{3}$$

where $P_W^{\tau,t}$ denotes the Walsh (1901) price index calculated for products from the set $G_{\tau,t}$.

3 Semi GEKS indices

New GEKS-type price indices have recently appeared in the literature that have broken the canons associated with assumptions towards the underlying index, located in the body of the GEKS index. These GEKS-type indices are not based on a superlative price index at all, nor on an index that meets the *time reversal test*. In Białek (2022*a*, 2022*b*) a general class of such indices (denoted by **GS-GEKS**) is proposed and its two special cases, i.e. the GEKS-L and GEKS-GL index, are discussed (see Sections 3.1 and 3.2). Without going into detail here, the papers cited prove that, with the appropriate assumptions to the base formula met, indices from the general class of semi-GEKS methods (**GS-GEKS**) satisfy many valuable tests (including the *identity test*).

3.1 The GEKS-L index

Let us denote the Laspeyres (1871) price index formula which compares the period s to the period τ on the basis of data from $G_{\tau,s}$ by $P_L^{\tau,s}$, i.e.:

$$P_L^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} q_i^{\tau} p_i^{s}}{\sum_{i \in G_{\tau,s}} q_i^{\tau} p_i^{\tau}}. \tag{4}$$

Under the above notation, the GEKS-L index can be defined as follows (Białek 2022*b*):

$$P_{GEKS-L}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_L^{\tau,t}}{P_L^{\tau,0}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{t}}{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{\tau}}}{\frac{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{0}}{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{\tau}}}\right)^{\frac{1}{T+1}}. \tag{5}$$

Please note that the GEKS-L index can be treated as the generalization of the Fisher price index formula ($P_F^{0,t}$) to the multi-period case. In fact, in a static item universe G observed over the two period time interval $[0,1]$, we obtain

$$P_{GEKS-L}^{0,1} = \prod_{\tau=0}^{1} \left(\frac{\sum_{i \in G} q_i^{\tau} p_i^{1}}{\sum_{i \in G} q_i^{\tau} p_i^{0}}\right)^{\frac{1}{1+1}} = \left(\frac{\sum_{i \in G} q_i^{0} p_i^{1}}{\sum_{i \in G} q_i^{0} p_i^{0}} \times \frac{\sum_{i \in G} q_i^{1} p_i^{1}}{\sum_{i \in G} q_i^{1} p_i^{0}}\right)^{\frac{1}{2}} = P_F^{0,1}, \tag{6}$$

since $G_0 = G_1 = G_{0,1} = G$.

3.2 The GEKS-GL index

Let us denote the geometric Laspeyres price index formula which compares the period s to the period τ on the basis of data from $G_{\tau,s}$ by $P_{GL}^{\tau,s}$, i.e.:

$$P_{GL}^{\tau,s} = \prod_{i \in G_{\tau,s}} \left(\frac{p_i^{s}}{p_i^{\tau}}\right)^{w_i^{\tau,s}(\tau)}, \tag{7}$$

where

$$w_i^{\tau,s}(\tau) = \frac{q_i^\tau p_i^\tau}{\sum_{k \in G_{\tau,s}} q_k^\tau p_k^\tau}. \tag{8}$$

Under the above notation, the GEKS-GL index can be defined as follows (Białek 2022*b*):

$$P_{GEKS-GL}^{0,t} = \prod_{\tau=0}^{T} (\frac{P_{GL}^{\tau,t}}{P_{GL}^{\tau,0}})^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} (\frac{\prod_{i \in G_{\tau,t}} (\frac{p_i^t}{p_i^\tau})^{w_i^{\tau,t}(\tau)}}{\prod_{i \in G_{\tau,0}} (\frac{p_i^0}{p_i^\tau})^{w_i^{\tau,0}(\tau)}})^{\frac{1}{T+1}}. \tag{9}$$

Please note that the GEKS-GL index can be treated as the generalisation of the Törnqvist (Törnqvist (1936)) price index formula ($P_T^{0,t}$) to the multi-period case. In fact, in a static item universe G observed over the two period time interval $[0, 1]$, we obtain

$$\begin{aligned} P_{GEKS-GL}^{0,1} &= \prod_{\tau=0}^{1} (\frac{\prod_{i \in G} (\frac{p_i^1}{p_i^\tau})^{w_i^{\tau,1}(\tau)}}{\prod_{i \in G} (\frac{p_i^0}{p_i^\tau})^{w_i^{\tau,0}(\tau)}})^{\frac{1}{1+1}} = \prod_{\tau=0}^{1} (\prod_{i \in G} (\frac{p_i^1}{p_i^0})^{w_i(\tau)})^{\frac{1}{2}} \\ &= \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(0)}{2}} \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(1)}{2}} = \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(0)+w_i(1)}{2}} = P_T^{0,1}, \end{aligned} \tag{10}$$

since $G_0 = G_1 = G_{0,1} = G$, and consequently $w_i^{\tau,0}(\tau) = w_i^{\tau,1}(\tau) = w_i(\tau)$ for any τ.

In the cited paper (Białek (2022*b*)) the following theorem is proved:

Theorem 1 *The GEKS-L and GEKS-GL indices satisfy the transitivity, identity, multi period identity, responsiveness, continuity, positivity and normalization, commensurability, price proportionality, homogeneity in prices and homogeneity in quantitites tests.*

4 Quality adjusted GEKS-type indices

Two new multilateral indices, the structure of which may resemble the idea of the GEKS index at first glance, were proposed in Białek (2023) (these proposals were firstly discussed in Białek (2022*a*)). However, the structure of the base index of the proposed multilateral formulas differs completely from the adopted convention related to the application of the superlative index. Moreover, the calculation of the base index will require *quality adjusting*, which in turn is more like the Geary-Khamis index idea. In fact, the proposed indices are in a sense a hybrid approach, i.e. they constitute a bridge between the quality adjusted unit value method and the GEKS method.

4.1 The GEKS-AQU index

In the unit value concept, prices of homogeneous products are equal to the ratio of expenditure and quantity sold (International Labour Office 2004, Chessa et al. 2017). However, quantities of different products cannot be added together as in the case of homogeneous products. That is why the idea of quality-adjusted unit value assumes that prices p_i^s of different products $i \in G_s$ in month s are transformed into "quality-adjusted prices" $\frac{p_i^s}{v_i}$ and quantities q_i^s are converted into "common units" $v_i q_i^s$ by using a set of factors $v = \{v_i : i \in G_s\}$ (Chessa et al. 2017). Thus, the "classical" quality adjusted unit value $QUV_{G_s}^s$ of a set of products G_s in month s can be expressed as follows:

$$QUV_{G_s}^s = \frac{\sum_{i \in G_s} q_i^s p_i^s}{\sum_{i \in G_s} v_i q_i^s} \tag{11}$$

The term “Quality-adjusted unit value method” (QU method for short) was introduced by Chessa (2015, 2016). The QU method is a family of unit value based index methods and its general form can be expressed by the following ratio:

$$P_{QU}^{0,t} = \frac{QUV_{G_t}^t}{QUV_{G_0}^0} \tag{12}$$

In practice, consumer response to price changes can be delayed or even accelerated as consumers not only react to current price changes but also use their own "forecasts" or concerns about future price increases. For example, consumption of thermophilic (seasonal) fruit is likely to be higher in summer because they are cheaper than in winter, when the season is almost over. For instance, some interesting study on "unconventional" consumer behaviour, such as stocking and delayed quantity responses to price changes, and its impact on chain drift bias can be found in the paper by von Auer (2019). Since in practice we often observe prices and quantities that are not perfectly synchronised in time, the following form of the "asynchronous quality-adjusted unit value" is proposed:

$$AQUV_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}, \tag{13}$$

where τ is any period from the considered time interval $[0, T]$. Obviously it holds that $AQUV_{G_{s,s}}^{s,s} = QUV_{G_s}^s$. Let us define now the function $P^{\tau,s}(v, q^\tau, p^\tau, p^s)$ as follows:

$$P^{\tau,s}(v, q^\tau, p^\tau, p^s) = \frac{AQUV_{G_{\tau,s}}^{\tau,s}}{AQUV_{G_{\tau,\tau}}^{\tau,\tau}}. \tag{14}$$

Putting (14) in the GEKS formula we obtain:

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}{\frac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{15}$$

Please note that the proposed index behaves like a GEKS index based on the Laspeyres index in the case of static item universe G. In fact, if the item universe is static, we obtain

$$\begin{aligned} P^{0,t}_{GEKS-AQU} &= \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i\in G} q_i^{\tau} p_i^t}{\sum_{i\in G} v_i q_i^{\tau}}}{\frac{\sum_{i\in G} q_i^{\tau} p_i^0}{\sum_{i\in G} v_i q_i^{\tau}}} \right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} \left(\frac{\sum_{i\in G} q_i^{\tau} p_i^t}{\sum_{i\in G} q_i^{\tau} p_i^0} \right)^{\frac{1}{T+1}} \\ &= \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i\in G} q_i^{\tau} p_i^t}{\sum_{i\in G} q_i^{\tau} p_i^{\tau}}}{\frac{\sum_{i\in G} q_i^{\tau} p_i^0}{\sum_{i\in G} q_i^{\tau} p_i^{\tau}}} \right)^{\frac{1}{T+1}} = P^{0,t}_{GEKS-L}. \end{aligned} \tag{16}$$

Finally, please also note, that theoretically the class of the $GEKS - AQU$ indices is infinite, since different choices of v_i factors lead to different index values. We could, for instance, consider v_i factors defined in the Geary-Khamis multilateral index resulting in a new, hybrid index, which would be a mixture of the GEKS and Geary-Khamis ideas. That would, however, be probably a slow solution. In this paper, we adopt the system of weights v_i corresponding to the augmented Lehr index (Lamboray 2017, van Loon & Roels 2018), where

$$v_i = \frac{\sum_{t=0}^{T} p_i^t q_i^t}{\sum_{t=0}^{T} q_i^t}. \tag{17}$$

4.2 The GEKS-AQI index

Let us note that formula (13) can be expressed by using quality-adjusted prices and quantities:

$$AQUV^{\tau,s}_{G_{\tau,s}} = \frac{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau} \frac{p_i^s}{v_i}}{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau}}. \tag{18}$$

If we replace all the adjusted prices ($\frac{p_i^s}{v_i}$) with the relative prices ($\frac{p_i^s}{p_i^{\tau}}$), then we obtain an "asynchronous quality-adjusted price index" (AQI), i.e.

$$AQI^{\tau,s}_{G_{\tau,s}} = \frac{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau} \frac{p_i^s}{p_i^{\tau}}}{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau}}. \tag{19}$$

This means that the AQI formula can be treated as a weighted arithmetic mean of partial indices $\frac{p_i^s}{p_i^{\tau}}$, where the weights are proportional to the relative share of the product's adjusted quantities (from the base period τ) in the sum of all adjusted quantities.

After inserting (19) into the GEKS formula formula we obtain:

$$P^{0,t}_{GEKS-AQI} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i\in G_{\tau,t}} v_i q_i^{\tau} \frac{p_i^t}{p_i^{\tau}}}{\sum_{i\in G_{\tau,t}} v_i q_i^{\tau}}}{\frac{\sum_{i\in G_{\tau,0}} v_i q_i^{\tau} \frac{p_i^0}{p_i^{\tau}}}{\sum_{i\in G_{\tau,0}} v_i q_i^{\tau}}} \right)^{\frac{1}{T+1}}. \tag{20}$$

Note that the GEKS-AQI index takes into account prices and quantities directly from all time window periods, while the GEKS-AQU index takes into account all quantities but only prices from the reference and base period. However, both formulas indirectly need information about the prices (and quantities) of products from each period in the time window to determine the factors v_i defined by formula (17). In this way, each new product in the analysed time window has an impact on the final value of the proposed indices.

The following theorem can be proved (Białek (2023)):

Theorem 2 *The GEKS-AQU and GEKS-AQI indices (15) satisfy the following tests: the transitivity,* ***identity****, multi period identity, responsiveness, continuity, positivity and normalisation, price proportionality and weak commensurability. If the item universe is the same in the compared periods* 0 *and* t *then the GEKS-AQU and GEKS-AQI indices satisfy also the homogeneity in prices and homogeneity in quantities tests.*

5 Generalizations of the GEKS method

Sections 5.2, 5.1 and 5.3 present proposals for certain generalizations of the GEKS index whose constructs have their genesis in the economic approach. That is why these proposals will be preceded by a brief introduction containing basic concepts and definitions related to the Cost of Living Index (COLI). It should be noted that considerations regarding the following general classes of indices have not yet been published anywhere and are currently under review in an economics journal.

The theory of the Cost of Living Index (COLI) for a single consumer (or household) was first developed by the Russian economist, Konüs (1924). In his economic approach, the period τ quantity vector $q^\tau = [q_1^\tau, q_2^\tau, ..., q_n^\tau]$ is determined by the consumer's preference function f and the period τ price vector $p^\tau = [p_1^\tau, p_2^\tau, ..., p_n^\tau]$ that the consumer faces observing n commodities or items. The consumer's preferences over the given consumption vector q are assumed to be represented by a continuous, non-decreasing and concave utility function f (International Labour Office 2004). The main assumption here is that the consumer minimises the cost of achieving the period τ utility level $u^\tau \equiv f(q^\tau)$ for the base period 0 and the current period t and consequently the consumer's cost function is defined as follows:

$$C(u^\tau, p^\tau) = min_q\{\sum_{i=1}^{n} p_i^\tau q_i : f(q) = u^\tau\}, \tau = 0, t. \tag{21}$$

The Konüs (1924) family of true cost of living indices comparing the current period with the base one, is defined as the ratio of minimum costs of achieving the same utility level $u \equiv f(q)$:

$$P_K(p^0, p^t, q) = \frac{C(f(q), p^t)}{C(f(q), p^0)}, \tag{22}$$

where $q = (q_1, q_2, ..., q_n)$ is a positive reference quantity vector. The **homothetic preferences** assumption simplifies the family of true cost of living indices as follows (International Labour Office 2004):

$$P_K(p^0, p^t, q) = \frac{c(p^t)}{c(p^0)}, \tag{23}$$

where $c(p) \equiv C(1, p)$ denotes the **unit cost function**.

If the $c(p)$ function has a **flexible functional form**, i.e. it can approximate an arbitrary twice differentiable linearly homogeneous cost function to the second order, then we call the index being exact for the $c(p)$ function **superlative** (Diewert 1976). It can be shown that the following quadratic mean of order r price index:

$$P^r(p^0, p^t, q^0, q^t) = \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 (\frac{p_i^t}{p_i^0})^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^t (\frac{p_i^t}{p_i^0})^{-r/2}}}. \tag{24}$$

is a **superlative** price index formula for any $r \neq 0$ (Diewert 1976, International Labour Office 2004).

The *implicit* quadratic mean of order r price index:

$$P_{im}^r(p^0, p^t, q^0, q^t) = \frac{\sum_{i=1}^{n} p_t^t q_i^t}{Q^r(p^0, p^t, q^0, q^t) \sum_{i=1}^{n} p_i^0 q_i^0}, \tag{25}$$

where the quadratic mean of order r quantity index Q^r can be written as follows:

$$Q^r(p^0, p^t, q^0, q^t) = \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 (\frac{q_i^t}{q_i^0})^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^t (\frac{q_i^t}{q_i^0})^{-r/2}}}, \tag{26}$$

is also a **superlative** price index formula (Diewert 1976, International Labour Office 2004). We have: $P_{im}^1 = P_W$, $P_{im}^2 = P_F$, $P^2 = P_F$ and $P^r(r \to 0) = P_T$, where P_W, P_F and P_T denote the superlative Walsh (1901), Fisher (1922) and Törnqvist (1936) price index respectively.

5.1 The GEKS-IQM index class

Let us define a general GEKS-type index family as follows:

$$P_{GEKS-IQM}^{0,t}(r) = \prod_{\tau=0}^{T} \left(\frac{P_{IQM}^{\tau,t}(r)}{P_{IQM}^{\tau,0}(r)} \right)^{\frac{1}{T+1}}, \tag{27}$$

where $P_{IQM}^{\tau,t}(r) \equiv P_{im}^r(p^\tau, p^t, q^\tau, q^t)$.

The IQM indices satisfy the *time reversal test*:

$$P_{IQM}^{0,t}(r) P_{IQM}^{t,0}(r) = \frac{\sum_{i \in G_{0,t}} p_i^t q_i^t}{Q^r(p^0,p^t,q^0,q^t) \sum_{i \in G_{0,t}} p_i^0 q_i^0} \frac{\sum_{i \in G_{0,t}} p_i^0 q_i^0}{Q^r(p^t,p^0,q^t,q^0) \sum_{i \in G_{0,t}} p_i^t q_i^t} = \frac{1}{Q^r(p^0,p^t,q^0,q^t) Q^r(p^t,p^0,q^t,q^0)} =$$
$$\frac{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^t (\frac{q_i^t}{q_i^0})^{-r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^0 (\frac{q_i^t}{q_i^0})^{r/2}}} \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^0 (\frac{q_i^0}{q_i^t})^{-r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^t (\frac{q_i^0}{q_i^t})^{r/2}}} = \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^t (\frac{q_i^0}{q_i^t})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^0 (\frac{q_i^t}{q_i^0})^{r/2}}} \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^0 (\frac{q_i^t}{q_i^0})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s_i^t (\frac{q_i^0}{q_i^t})^{r/2}}} = 1.$$

It can be proved that the following Theorem holds:

Theorem 3 *For each $r \neq 0$ the multilateral price index $P^{0,t}_{GEKS-IQM}(r)$ satisfies the following tests: the transitivity, multi period identity, responsiveness, continuity, positivity and normalisation, homogeneity in prices and homogeneity in quantities, as well as commensurability.*

We may obtain some known particular cases of this index class:

$$P^{0,t}_{GEKS-IQM}(1) = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}_{W}}{P^{\tau,0}_{W}} \right)^{\frac{1}{T+1}} = P^{0,t}_{GEKS-W} \tag{28}$$

$$P^{0,t}_{GEKS-IQM}(2) = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}_{F}}{P^{\tau,0}_{F}} \right)^{\frac{1}{T+1}} = P^{0,t}_{GEKS}. \tag{29}$$

$$P^{0,t}_{GEKS-IQM}(r \rightarrow 0) \approx \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}_{T}}{P^{\tau,0}_{T}} \right)^{\frac{1}{T+1}} = P^{0,t}_{CCDI}. \tag{30}$$

5.2 The GEKS-QM index class

Let us define a general GEKS-type index family as follows:

$$P^{0,t}_{GEKS-QM}(r) = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}_{QM}(r)}{P^{\tau,0}_{QM}(r)} \right)^{\frac{1}{T+1}}. \tag{31}$$

where $P^{\tau,t}_{QM}(r) \equiv P^{r}(p^{\tau}, p^{t}, q^{\tau}, q^{t})$.

The QM indices satisfy the *time reversal test*:

$$P^{0,t}_{QM}(r) P^{t,0}_{QM}(r) = \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s^{0}_{i} (\frac{p^{t}_{i}}{p^{0}_{i}})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s^{t}_{i} (\frac{p^{t}_{i}}{p^{0}_{i}})^{-r/2}}} \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s^{t}_{i} (\frac{p^{0}_{i}}{p^{t}_{i}})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s^{0}_{i} (\frac{p^{0}_{i}}{p^{t}_{i}})^{-r/2}}} =$$

$$= \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s^{0}_{i} (\frac{p^{t}_{i}}{p^{0}_{i}})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s^{t}_{i} (\frac{p^{0}_{i}}{p^{t}_{i}})^{r/2}}} \frac{\sqrt[r]{\sum_{i \in G_{0,t}} s^{t}_{i} (\frac{p^{0}_{i}}{p^{t}_{i}})^{r/2}}}{\sqrt[r]{\sum_{i \in G_{0,t}} s^{0}_{i} (\frac{p^{t}_{i}}{p^{0}_{i}})^{r/2}}} = 1.$$

It can be proved that the following Theorem holds:

Theorem 4 *For each $r \neq 0$ the multilateral price index $P^{0,t}_{GEKS-QM}(r)$ satisfies the following tests: the transitivity, multi period identity, responsiveness, continuity, positivity and normalisation, weak price proportionality, homogeneity in prices and homogeneity in quantities tests, as well as commensurability.*

We may obtain some known particular cases of this index class:

$$P^{0,t}_{GEKS-QM}(r \rightarrow 0) = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}_{T}}{P^{\tau,0}_{T}} \right)^{\frac{1}{T+1}} = P^{0,t}_{CCDI}, \tag{32}$$

$$P_{GEKS-QM}^{0,t}(1) \approx \prod_{\tau=0}^{T} \left(\frac{P_W^{\tau,t}}{P_W^{\tau,0}} \right)^{\frac{1}{T+1}} = P_{GEKS-W}^{0,t}, \tag{33}$$

$$P_{GEKS-QM}^{0,t}(2) = \prod_{\tau=0}^{T} \left(\frac{P_F^{\tau,t}}{P_F^{\tau,0}} \right)^{\frac{1}{T+1}} = P_{GEKS}^{0,t}. \tag{34}$$

5.3 The GEKS-LM index class

The Lloyd-Moulton price index (Loyd 1975) can be written as follows:

$$P_{LM}^{\tau,t}(\sigma) = (\sum_{i \in G_{\tau,t}} s_i^{\tau} (\frac{p_i^t}{p_i^{\tau}})^{1-\sigma})^{\frac{1}{1-\sigma}}, \tag{35}$$

where the parameter σ denotes the elasticity of substitution. It can be shown (Loyd 1975, Moulton 1996) that under the assumption of cost minimising the behaviour the index P_{LM} is exact for CES preferences, i.e. for the fixed periods τ and t, it holds that

$$P_{LM}^{\tau,t}(\sigma) = \frac{c_{\sigma}(p^t)}{c_{\sigma}(p^{\tau})}, \tag{36}$$

where the unit cost function $c_{\sigma}(p)$ is a constant elasticity of substitution (CES) aggregator function introduced by Arrow et al. (1961).

On the bases of the Lloyd-Moulton index, let us define a general GEKS-type index family as follows:

$$P_{GEKS-LM}^{0,t}(\sigma) = \prod_{\tau=0}^{T} \left(\frac{P_{LM}^{\tau,t}(\sigma)}{P_{LM}^{\tau,0}(\sigma)} \right)^{\frac{1}{T+1}}, \tag{37}$$

with the following special cases:

$$P_{GEKS-LM}^{0,t}(0) = \prod_{\tau=0}^{T} \left(\frac{P_{La}^{\tau,t}}{P_{La}^{\tau,0}} \right)^{\frac{1}{T+1}} = P_{GEKS-L}^{0,t} \tag{38}$$

and

$$P_{GEKS-LM}^{0,t}(\sigma \rightarrow 1) = \prod_{\tau=0}^{T} \left(\frac{P_{GLa}^{\tau,t}}{P_{GLa}^{\tau,0}} \right)^{\frac{1}{T+1}} = P_{GEKS-GL}^{0,t}. \tag{39}$$

It can be proved that the following theorem holds:

Theorem 5 *For each $\sigma \neq 1$ the multilateral price index $P_{GEKS-LM}^{0,t}(\sigma)$ satisfies the transitivity,* ***identity****, multi period identity, responsiveness, continuity, positivity and normalisation, commensurability, strong price proportionality, as well as homogeneity in prices and homogeneity in quantities tests.*

Remark 1. Please note that we make a distinction between *strong* and *weak price proportionality test* in the paper. To the best of the author's knowledge, this is the first distinction of this type in the literature. It is worth emphasizing, however, that it only makes sense in the case of multilateral indices, where the considerations concern many periods within a given time window. The *weak price proportionality* assumes that price changes in all periods from the time interval are proportional to prices from the base period. This test states that in such a situation, the index comparing the current and base periods should depend only on the proportion of prices from the compared periods. In the case of a strong version of this test (i.e. *strong proportionality test*), the same behaviour of the index is expected under a less restrictive requirement, because it assumes proportionality of prices in only two compared periods (prices from periods earlier than the current one do not have to be proportional to prices from the base period). Please note, that in the case of a bilateral approach, the distinction between weak and strong price proportionality would not exist because no intermediate periods between the base period and the current period are then considered. It is easy to verify that a strong version of this test implies the occurrence of a weak version of the test. It can be shown that the GEKS, Geary-Khamis (Geary 1958, Khamis 1972), or TPD (de Haan & Krsinich 2018) multilateral indices satisfy the *weak price proportionality test* but do not satisfy the *strong price proportionality*, while the SPQ (Diewert 2020), GEKS-L, or GEKS-GL indices satisfy the *strong price proportionality*.

Remark 2. In the author's opinion, there is no point in considering the *identity test, multi period identity test* and *weak* or *strong proportionality test* if we do not assume that the product universe is the same in the base and current periods. Of course, such an assumption does not exclude the phenomenon of product churn inside the time interval under consideration. Zhang et al. (2019) had a similar comment. In the cited paper, for example, we read on page 689: "In order to verify whether or not chain drifting is the case, one must compare the chained index to the direct index that could have been calculated between 0 and t (i.e., between the base and current periods). (...) To push the difficulty to the logical extreme, suppose the item universe is completely renewed. What are the conditions of non-drifting, or transitivity, in this case?"

6 Empirical illustration

An empirical illustration of the potential differences between the proposed indices (including representatives of general index classes) and the multilateral indices known from the literature (GEKS, CCDI, TPD, Geary-Khamis) was carried out on the **milk** and **coffee** datasets. These datasets are anonymized sets of real data sets on coffee and milk sales in one of the retail chains operating in Poland. It should be noted that these datasets are available in the R package *PriceIndices* (Białek 2021) and all results presented here are reproducible (the corresponding program code is included in Appendix A). The analysis was performed at the GTIN code level and for the period Dec, 2018 - Dec, 2019. The product rotation corresponding to these datasets and the analyzed time period is presented in Fig. 1. Fig. 2 compares indices presented in Sections 3.1, 3.2, 4.1 and 4.2 with the popular multilateral indices. An analogous comparison, except that concerning indices from the GEKS-IQM class, is shown in Fig. 3. The next two figures (i.e. Fig. 4 and Fig. 5) consider the other two general classes of indices, i.e. the GEKS-QM and GEKS-LM index classes. The corresponding scripts, written in R language by using the *PriceIndices* package, can be found in Appendix A.

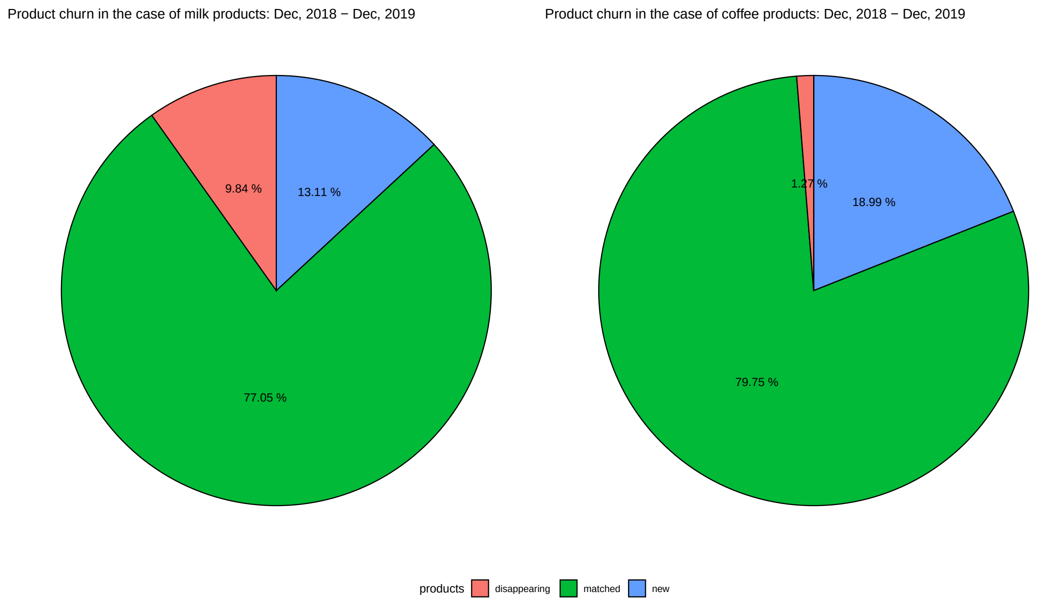

Figure 1: Product churn in the considered scanner data sets.

Figure 2: Comparison of the GEKS-L, GEKS-GL, GEKS-AQI and GEKS-AQU indices with known multilateral indices.

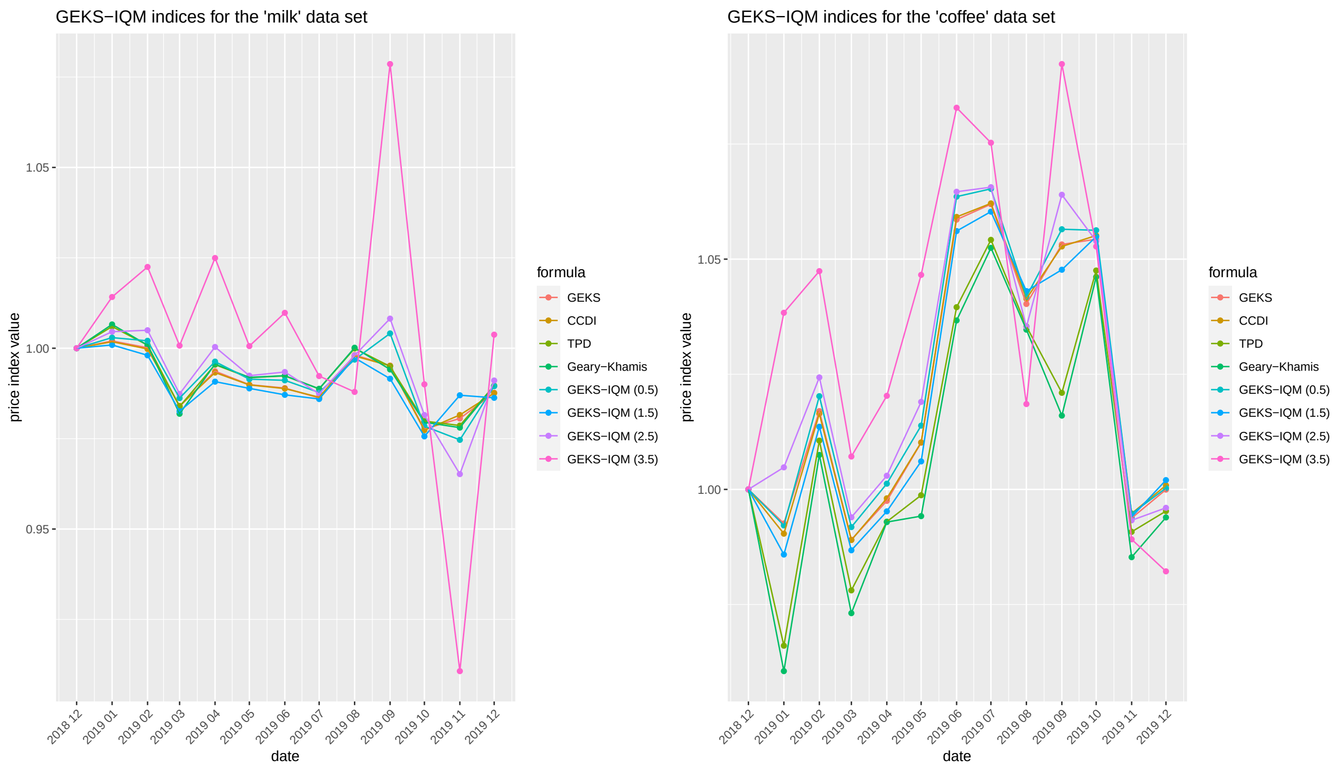

Figure 3: Comparison of the selected GEKS-IQM indices with known multilateral indices.

Figure 4: Comparison of the selected GEKS-QM indices with known multilateral indices.

Figure 5: Comparison of the selected GEKS-LM indices with known multilateral indices.

7 Conclusions

Let us start first by analyzing the empirical example presented in Section 6. Observing Fig. 2, one can see that, in general, the multilateral indices determined on the basis of the *milk* and *coffee* data sets, form three clusters of values. Namely, the values of the GEKS-L, GEKS-GL, GEKS-AQI and GEKS-AQU indices are close to each other (**Cluster 1**), further - the GEKS and CCDI indices approximate each other (**cluster 2**), and the TPD and Geary-Khamis indices have similar values (**Cluster 3**). In our study, it turned out that values of indices from the **Cluster 3** are always between index values from **Clusters 1** and **Cluster 2**.

Fig. 3 casts a bit of a shadow over the usefulness of indices from the GEKS-IQM class, as they begin to unreasonably lag behind typical multilateral indices for the parameter $r > 2$. On the other hand, indices from the GEKS-QM class are stable due to the value of the parameter r (Fig. 4). At the same time, differences between these indices and typical multilateral ones can be both small (no more than 0.5 p.p. as in the case of the *milk* set) and noticeable (as much as 2.5 p.p. in the case of the *coffee* set). Fig. 5 shows that also the indices from the GEKS-LM class form a separate cluster of values and they are moderately sensitive to the choice of σ parameter. In the case of the *coffee* data set, the indices from this class have substantially bigger values than the other compared indices, with differences exceeding even 3 p.p. In the case of the *milk* data set, indices from the GEKS-LM class generate the lowest values among the set of multilateral indices considered.

From a theoretical (axiomatic) point of view, the GEKS-LM class of indices seems to be the most valuable (e.g., they satisfy the *identity test*). Important special cases of this class are the GEKS-L and GEKS-GL indices. In order to satisfy the axiomatic and economic approaches simultaneously, it would be necessary to determine the elasticity of substitution parameter beforehand (e.g. by using a panel regression model explaining the behavior of quantities based on prices). This is the direction of the author's further research.

Please also note that indices from the GEKS-AQU and GEKS-AQI classes also satisfy the *identity test*. However, a certain limitation of indices from these classes is the condition necessary to satisfy *homogeneity in prices* and *homogeneity in quantities*. This condition is a less realistic scenario, implying a constant set of products in the base and current periods being compared.

Appendix A R scripts for figures presented in Section 6

```r
library("PriceIndices")
library("ggplot2")
library("ggpubr")

## Loading required package: magrittr

df1<-price_indices(milk,
                  start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                  "geksl","geksgl","geksaqu","geksaqi"),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                  "GEKS-L","GEKS-GL",
                  "GEKS-AQU","GEKS-AQI"),
                  interval = TRUE)
fig1<-compare_indices_df(df1)
fig1<-fig1+
ggtitle("GEKS-type indices for the 'milk' data set")


df2<-price_indices(coffee,
                  start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                  "geksl","geksgl","geksaqu","geksaqi"),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                  "GEKS-L","GEKS-GL",
                  "GEKS-AQU","GEKS-AQI"),
                  interval = TRUE)


fig2<-compare_indices_df(df2)
fig2<-fig2+

ggtitle("GEKS-type indices for the 'coffee' data set")


fig<-ggarrange(plotlist=list(fig1, fig2),ncol=2,nrow=1)
ggsave(filename="fig2.pdf",plot=fig, width = 14, height=8)
```

Figure 6: R script that allows to create a Fig. 2.

```r
library("PriceIndices")
library("ggplot2")
library("ggpubr")

df1<-price_indices(milk, start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                            "geksiqm","geksiqm","geksiqm","geksiqm"),
                  r=c(0.5, 1.5, 2.5, 3.5),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                          "GEKS-IQM (0.5)","GEKS-IQM (1.5)",
                          "GEKS-IQM (2.5)","GEKS-IQM (3.5)"),
                  interval = TRUE)


fig1<-compare_indices_df(df1)


fig1<-fig1+ggtitle("GEKS-IQM indices for the 'milk' data set")

df2<-price_indices(coffee, start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                            "geksiqm","geksiqm","geksiqm","geksiqm"),
                  r=c(0.5, 1.5, 2.5, 3.5),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                          "GEKS-IQM (0.5)","GEKS-IQM (1.5)",
                          "GEKS-IQM (2.5)","GEKS-IQM (3.5)"),
                  interval = TRUE)


fig2<-compare_indices_df(df2)


fig2<-fig2+ggtitle("GEKS-IQM indices for the 'coffee' data set")
```

```r
fig<-ggarrange(plotlist=list(fig1, fig2),ncol=2,nrow=1)
ggsave(filename="fig3.pdf",plot=fig, width = 14, height=8)
```

Figure 7: R script that allows to create a Fig. 3.

```r
library("PriceIndices")
library("ggplot2")
library("ggpubr")

df1<-price_indices(milk, start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                            "geksqm","geksqm","geksqm","geksqm"),
                  r=c(0.5, 1.5, 2.5, 3.5),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                          "GEKS-QM (0.5)","GEKS-QM (1.5)",
                          "GEKS-QM (2.5)","GEKS-QM (3.5)"),
                  interval = TRUE)
fig1<-compare_indices_df(df1)
fig1<-fig1+ggtitle("GEKS-QM indices for the 'milk' data set")

df2<-price_indices(coffee, start = "2018-12", end = "2019-12",
                  formula=c("geks", "ccdi", "tpd","gk",
                            "geksqm","geksqm","geksqm","geksqm"),
                  r=c(0.5, 1.5, 2.5, 3.5),
                  window=replicate(8,13),
                  names=c("GEKS","CCDI","TPD","Geary-Khamis",
                          "GEKS-QM (0.5)","GEKS-QM (1.5)",
                          "GEKS-QM (2.5)","GEKS-QM (3.5)"),
                  interval = TRUE)
fig2<-compare_indices_df(df2)
fig2<-fig2+ggtitle("GEKS-QM indices for the 'coffee' data set")
fig2
```

```r
fig<-ggarrange(plotlist=list(fig1, fig2),ncol=2,nrow=1)
ggsave(filename="fig4.pdf",plot=fig, width = 14, height=8)
```

Figure 8: R script that allows to create a Fig. 4.

```r
library("PriceIndices")
library("ggplot2")
library("ggpubr")
```

```r
df1<-price_indices(milk, start = "2018-12", end = "2019-12",
                   formula=c("geks", "ccdi", "tpd","gk",
                             "gekslm","gekslm","gekslm","gekslm"),
                   sigma=c(0.5, 1.5, 2.5, 3.5),
                   window=replicate(8,13),
                   names=c("GEKS","CCDI","TPD","Geary-Khamis",
                           "GEKS-LM (0.5)","GEKS-LM (1.5)",
                           "GEKS-LM (2.5)","GEKS-LM (3.5)"),
                   interval = TRUE)
fig1<-compare_indices_df(df1)
fig1<-fig1+ggtitle("GEKS-LM indices for the 'milk' data set")

df2<-price_indices(coffee, start = "2018-12", end = "2019-12",
                   formula=c("geks", "ccdi", "tpd","gk",
                             "gekslm","gekslm","gekslm","gekslm"),
                   sigma=c(0.5, 1.5, 2.5, 3.5),
                   window=replicate(8,13),
                   names=c("GEKS","CCDI","TPD","Geary-Khamis",
                           "GEKS-LM (0.5)","GEKS-LM (1.5)",
                           "GEKS-LM (2.5)","GEKS-LM (3.5)"),
                   interval = TRUE)
fig2<-compare_indices_df(df2)
fig2<-fig2+ggtitle("GEKS-LM indices for the 'coffee' data set")
fig2
```

```r
fig<-ggarrange(plotlist=list(fig1, fig2),ncol=2,nrow=1)
ggsave(filename="fig5.pdf",plot=fig, width = 14, height=8)
```

Figure 9: R script that allows to create a Fig. 5.

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