Price Indexes for Elementary Aggregates: the Sampling Approach

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Abstract: At the lowest level of aggregation of a CPI or PPI quantity information is usually unavailable and nothing but matched samples of prices are used for the index computation. Familiar indexes used at this level of aggregation are those of Dutot, Carli, and Jevons. An important, yet often overlooked characteristic of these and similar indexes is that they are sample statistics, whose properties can be studied from the sampling point of view. This paper provides a systematic study of this topic and concludes with a number of recommendations for statistical practice.

Keywords: CPI, PPI, elementary aggregate, price index.

1. Introduction

Mainstream (bilateral) index number theory applies to aggregates consisting of a finite set of commodities. Two basic assumptions are that the set of commodities does not change between the two periods compared, and that all the price and quantity data which are necessary for the computation of an index are available to the statistician. In this paper we are concerned with what to do when the second of these assumptions is or cannot be fulfilled. There are, of course, various kinds of unavailability of data. The situation we will consider in particular in this paper is that nothing but price data are available for a sample of commodities and/or respondents.

Since such a situation materializes at the very first stage of the computation of any official price index, such as a Consumer Price Index (CPI) or a Producer Price Index (PPI), we are dealing here with an issue of great practical significance.

The usual approach to the problem of unavailable quantity data is to consider price indexes which are functions of prices only. The main formulas discussed in the literature and used in practice are

- the ratio of arithmetic average prices (the formula of Dutot),
- the arithmetic average of price relatives (the formula of Carli),
- the geometric average of price relatives = the ratio of geometric average prices (the formula of Jevons).

[1] The views expressed in this paper are those of the author and do not necessarily reflect any policy of Statistics Netherlands.

The appropriateness of these formulas has been studied by various methods. Following the early contribution of Eichhorn and Voeller (1976), Dalén (1992) and Diewert (1995) studied their properties from an axiomatic point of view. Additional insights were obtained by deriving (approximate) numerical relations between these formulas, and by combining these relations with more or less intuitive economic reasoning. Balk's (1994) approach was to see which assumptions would validate these formulas as estimators of true but unknown population price indexes, which by definition are functions of prices and quantities. A summary of the state of affairs, written from the CPI perspective, but easily generalizable to other perspectives, was recently provided by Diewert (2002).

This paper develops the sampling approach. In section 2 it is argued that, although not known to the statistician, all the detailed price and quantity data of the commodities and respondents pertaining to the aggregate under consideration exist in the real world. Section 3 then argues that the first task faced by the statistician is to decide on the nature of the aggregate (homogeneous or heterogeneous) and on the target price index (the unit value index or some superlative or non-superlative price index). Next the sampling design comes into the picture. With help of these two pieces of information, one can judge the various estimators with respect to their performance. This is the topic of section 4, which is on homogeneous aggregates, and sections 5 – 7, which are on heterogeneous aggregates and superlative target price indexes. Section 8 adds to this topic with some micro-economic considerations on the choice of a sample price index. Section 9 discusses the not unimportant case where, for operational reasons, a non-superlative price index was chosen as target. Section 10 surveys the behaviour of the various sample price indexes with respect to the Time Reversal Test, and reviews the (approximate) numerical relations between them. Section 11 summarizes the key results and concludes with practical advice.

2. Setting the stage

The aggregates covered by a CPI or a PPI are usually arranged in the form of a tree-like hierarchy (such as COICOP or NACE). Any *aggregate* is a set of economic transactions pertaining to a set of commodities. Commodities can be goods or services. Every economic transaction relates to the change of ownership (in the case of a good) or the delivery (in the case of a service) of a specific, well-defined commodity at a particular place and date, and comes with a quantity and a price. The price index for an aggregate is calculated as a weighted average of the price indexes for the subaggregates, the (expenditure or sales) weights and type of average being determined by the index formula. Descent in such a hierarchy is possible as far as available information allows the weights to be decomposed. The lowest level aggregates are called *elementary* aggregates. They are basically of two types:

- those for which all detailed price and quantity information is available;
- those for which the statistician, considering the operational cost and/or the response burden of getting detailed price and quantity information about all the transactions, decides to make use of a representative sample of commodities and/or respondents.

Any actual CPI or PPI, considered as a function which transforms sample survey data into an index number, can be considered as a stochastic variable, whose expectation ideally equals its population counterpart. The elementary aggregates then serve as strata for the sampling procedure. We are of course particularly interested in strata of the second type.

The practical relevance of studying this topic is large. Since the elementary aggregates form the building blocks of a CPI or a PPI, the choice of an inappropriate formula at this level can have a tremendous impact higher-up in the aggregation tree.

The detailed price and quantity data, although not available to the statistician, nevertheless – at least in principle – exist in the outside world. It is thereby frequently the case that at the respondent level (outlet or firm) already some aggregation of the basic transaction information has been executed, usually in a form that suits the respondent's financial and/or logistic information system. This could be called the basic information level. This is, however, in no way a naturally given level. One could always ask the respondent to provide more disaggregated information. For instance, instead of monthly data one could ask for weekly data; whenever appropriate, one could ask for regional instead of global data; or, one could ask for data according to a finer commodity classification. The only natural barrier to further disaggregation is the individual transaction level.[2]

Thus, conceptually, for all well-defined commodities belonging to a certain elementary aggregate and all relevant respondents there exists information on both the quantity sold and the associated average price (unit value) over a certain time period. Let us try to formalize this somewhat. The basic information – which in principle exists in the outside world – is of the form $\{(p_n^t, q_n^t); n = 1,..., N\}$ where t denotes a time period, the elements of the population of (non-void) pairs of well-defined commodities and respondents, henceforth called elements, are labelled from 1 to N, p_n^t denotes the price and q_n^t denotes the quantity of element n at time period t. It may be clear that N may be a very large number, since even at very low levels of aggregation there can be thousands of elements involved. We repeat that it will be assumed that the population does not change between the time periods considered.

It is assumed that we must compare a later period 1 to an earlier period 0. The later period will be called comparison period and the earlier period base period. The conceptual problem is to split the value change

$$(1)\quad \sum_{n=1}^{N} p_n^1 q_n^1 \Big/ \sum_{n=1}^{N} p_n^0 q_n^0$$

multiplicatively into a price index and a quantity index. This is traditionally called the index number problem. Both indexes should depend only on the prices and quantities of the two periods.

3. Homogeneity or heterogeneity

There is now an important conceptual choice to make. In the statistician's parlance this is known as the 'homogeneity or heterogeneity' issue. Although in the literature lots of words have been devoted to this issue, at the end of the day the whole problem can be reduced to the rather simple looking operational question:

(2) Does it make (economic) sense to add up the quantities q_n^t of the elements n=1,…,N ?

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[2] See Balk (1994) for a similar approach.

If the answer to this question is ‘yes’, then the appropriate, also called target, price index for the elementary aggregate is the unit value index

$$(3)\quad P_U \equiv \frac{\sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} q_n^1}{\sum_{n=1}^{N} p_n^0 q_n^0 / \sum_{n=1}^{N} q_n^0},$$

that is, the average comparison period price divided by the average base period price.[3] The corresponding quantity index is the simple sum or Dutot index

$$(4)\quad Q_D \equiv \sum_{n=1}^{N} q_n^1 / \sum_{n=1}^{N} q_n^0 .$$

When the quantities are additive, we are obviously dealing with a situation where a certain commodity at the same time is sold or bought at different places at different prices. Put otherwise, we are dealing with pure price differences. These can be caused by market imperfections, such as price discrimination, consumer ignorance, or rationing. Economic theory seems to preclude this possibility since it states that in equilibrium “the law of one price” must hold. In reality, however, market imperfections are the rule rather than the exception. But also physical restrictions can play a role. Although, for instance, the “representative” consumer is assumed to be fully informed about all prices and to have immediate and costless access to all outlets throughout the country, the sheer physical distance between the outlets precludes “real” consumers from exploiting this magical possibility; thus, price differences exist where they, according to a representative-agent-based theory, are not supposed to exist.

If the answer to question (2) is ‘no’, then there are various options available for the target price index. First of all, the axiomatic as well as the economic approach to index number theory leads to the conclusion that the target price index should be some superlative index. According to the recent survey by Diewert (2002; Section 5), three price indexes appear to be particularly relevant. The first is the Törnqvist price index

$$(5)\quad P_T \equiv \prod_{n=1}^{N} (p_n^1 / p_n^0)^{(s_n^0 + s_n^1)/2},$$

where $s_n^t \equiv p_n^t q_n^t / \sum_{n=1}^{N} p_n^t q_n^t$ (t=0,1) is element n’s value share in period t. This price index is a weighted geometric average of the price relatives, the weights being average value shares. The corresponding quantity index is defined as

$$(6)\quad \tilde{Q}_T \equiv \left(\sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} p_n^0 q_n^0 \right) / P_T .$$

[3] Balk (1998) shows that the unit value index satisfies the conventional axioms for a price index, except the commensurability axiom and the proportionality axiom. However, when the elements are commensurate, the commensurability axiom reduces to $P(\lambda p^1, \lambda^{-1} q^1, \lambda p^0, \lambda^{-1} q^0) = P(p^1, q^1, p^0, q^0) (\lambda > 0)$, which clearly is satisfied.

The second superlative price index is the Fisher index,

$$(7)\quad P_F \equiv \left(\frac{\sum_{n=1}^{N} p_n^1 q_n^0}{\sum_{n=1}^{N} p_n^0 q_n^0} \right)^{1/2} \left(\frac{\sum_{n=1}^{N} p_n^1 q_n^1}{\sum_{n=1}^{N} p_n^0 q_n^1} \right)^{1/2} = (P_L P_P)^{1/2},$$

which is the geometric average of the Laspeyres and the Paasche price indexes. In this case the quantity index is given by

$$(8)\quad Q_F \equiv \left(\frac{\sum_{n=1}^{N} p_n^0 q_n^1}{\sum_{n=1}^{N} p_n^0 q_n^0} \right)^{1/2} \left(\frac{\sum_{n=1}^{N} p_n^1 q_n^1}{\sum_{n=1}^{N} p_n^1 q_n^0} \right)^{1/2} = (Q_L Q_P)^{1/2},$$

which is the geometric average of the Laspeyres and the Paasche quantity indexes. The third superlative price index is the Walsh index, defined as

$$(9)\quad P_W \equiv \frac{\sum_{n=1}^{N} p_n^1 (q_n^0 q_n^1)^{1/2}}{\sum_{n=1}^{N} p_n^0 (q_n^0 q_n^1)^{1/2}},$$

in which case the quantity index is defined by

$$(10)\quad \tilde{Q}_W \equiv \left(\sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} p_n^0 q_n^0 \right) / P_W .$$

The Walsh price index is a member of the class of so-called basket price indexes, that is, price indexes which compare the cost of a certain basket of quantities in the comparison period to the cost in the base period. The Laspeyres and Paasche price indexes are typical examples: the first employs the base period basket and the second the comparison period basket. The basket of the Walsh price index is an artificial one, namely consisting of the geometric averages of the quantities of the two periods.

Many statistical offices, however, are forced by operational reasons to define a non-superlative price index as target. In general their target appears to have the form of a Lowe price index

$$(11)\quad P_{Lo} \equiv \frac{\sum_{n=1}^{N} p_n^1 q_n^b}{\sum_{n=1}^{N} p_n^0 q_n^b},$$

where b denotes some period prior to the base period 0. The corresponding quantity index is then defined by

$$(12)\quad \tilde{Q}_{Lo} \equiv \left(\sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} p_n^0 q_n^0 \right) / P_{Lo} .$$

Notice that the five price indexes considered above all satisfy the Time Reversal test, that is, using obvious notation, $P(p^1, q^1, p^0, q^0) = 1 / P(p^0, q^0, p^1, q^1)$.

It could be that the statistician is unable to decide between a simple ‘yes’ or ‘no’ reply to (2), that is, he or she finds that for certain subsets of the elementary aggregate {1,…,N} it makes sense to add up the quantities whereas for the remainder this does not make sense. Then the aggregate should be split into subsets to which either the ‘yes’ or the ‘no’ answer applies. If this splitting appears to be not feasible then the ‘no’ answer should take precedence over the ‘yes’ answer. Thus, conceptually, we have to deal with but two cases.

Having defined the target price (and quantity) index, the statistician must face the basic problem that not all the information on the prices and quantities of the elements is available. The maximum he or she can obtain is information $\{(p_n^t, q_n^t); t = 0,1; n \in S\}$ for a sample $S \subset \{1,..., N\}$. More realistic, however, is the situation where the information set has the form $\{p_n^t; t = 0,1; n \in S\}$, that is, nothing but a matched sample of prices is available. From this sample information the population price index (or quantity index) must be estimated. This is the point where the theory of finite population sampling will appear to be helpful for obtaining insight into the properties of the various estimators.

At the outset we must notice that in practice the way the sample S is drawn usually remains hidden in a certain darkness. The main problem is that there is no such thing as a sampling frame. Knowledge about the composition of the elementary aggregate, in the form of an exhaustive listing of all its elements, is usually absent. There is only, more or less ad hoc, evidence available about particular elements belonging or not belonging to this aggregate. In order to use the theory of finite population sampling, however, we must make certain assumptions about the sampling design.

In the remainder of this paper we will consider two scenarios. Each of these is believed to be more or less representative of actual statistical practice. The first scenario assumes that S is a simple random sample, which means that each element of the population has the same probability of being included in the sample. This so-called (first order) inclusion probability is $\Pr(n \in S) = \varsigma(S)/N$, where $\varsigma(S)$ denotes the (prespecified) sample size.

In the second secenario the more important elements of the population have a correspondingly larger probability of being included in the sample than the less important elements. This will be formalized by assuming that the elements were drawn with probability proportional to size, where size denotes some measure of importance. If the size of element n is denoted by a positive scalar a_n (n=1,…,N), then the probability that element n is included in the sample S is $\Pr(n \in S) = \varsigma(S) a_n / \sum_{n=1}^{N} a_n$. Without loss of generality, it will be assumed that $\Pr(n \in S) < 1$ for n=1,…,N.[4] Notice that in both scenarios $\sum_{n=1}^{N} \Pr(n \in S) = \varsigma(S)$.

The general question we first consider is whether it is at all possible to find an estimator for P_U, P_T, P_F, or P_W which uses, in addition to base period and comparison period price information, nothing but base period quantity information. Within the first secenario, the answer is obviously ‘no’, because each of these target indexes depends also on comparison period quantity information, and this information is nowhere involved in the sampling design.

[4] Elements for which initially this probability would turn out to be greater than or equal to 1 are selected with certainty and from the remaining set of elements a sample is drawn.

There is no free lunch here. Put otherwise, any estimator which is based on the data set $\{p_n^0, q_n^0, p_n^1; n \in S\}$ will necessarily be biased. This conclusion obviously extends to any estimator which is based on the (smaller) data set $\{p_n^0, p_n^1; n \in S\}$. Within the second scenario, however, the answer depends on the extent to which comparison period quantity information can be assumed to be included in the size measure.

4. Homogeneous aggregates

Suppose we deal with a homogeneous aggregate. Then the target (or population) price index is the unit value index P_U. If the total base period value $\sum_{n=1}^{N} p_n^0 q_n^0$ as well as the total comparison period value $\sum_{n=1}^{N} p_n^1 q_n^1$ is known, the obvious route to take – see expression (3) – is to estimate the Dutot quantity index Q_D. A likely candidate is its sample counterpart

$$(13)\quad \hat{Q}_D \equiv \sum_{n \in S} q_n^1 / \sum_{n \in S} q_n^0 .$$

Suppose that S is a simple random sample and recall that the inclusion probabilities are $\Pr(n \in S) = \varsigma(S)/N$, where $\varsigma(S)$ denotes the sample size. Then the expected value of the sample Dutot quantity index is

$$(14)\quad E(\hat{Q}_D) = E\left(\frac{(1/\varsigma(S))\sum_{n \in S} q_n^1}{(1/\varsigma(S))\sum_{n \in S} q_n^0} \right) \approx \frac{E((1/\varsigma(S))\sum_{n \in S} q_n^1)}{E((1/\varsigma(S))\sum_{n \in S} q_n^0)} = \frac{(1/\varsigma(S))\sum_{n=1}^{N} q_n^1 \Pr(n \in S)}{(1/\varsigma(S))\sum_{n=1}^{N} q_n^0 \Pr(n \in S)}$$
$$= \frac{(1/N)\sum_{n=1}^{N} q_n^1}{(1/N)\sum_{n=1}^{N} q_n^0} = Q_D .$$

Expression (14) means that $\hat{Q}_D$ is an approximately unbiased estimator of the population Dutot quantity index Q_D. The bias[5] is of technical nature and will approach zero when the sample size gets larger.

Consider now the sample Carli quantity index, defined as

$$(15)\quad \hat{Q}_C \equiv \frac{1}{\varsigma(S)} \sum_{n \in S} (q_n^1 / q_n^0) .$$

Assume that the elements were drawn with probability proportional to size, whereby the size of element n is defined as its base period quantity share $q_n^0 / \sum_{n=1}^{N} q_n^0$ (n=1,…,N). Thus the probability that element n is included in the sample is equal to $\Pr(n \in S) = \varsigma(S) q_n^0 / \sum_{n=1}^{N} q_n^0$. Then the expected value of the sample Carli quantity index is equal to

[5] The bias is due to the fact that we approximate *E(x/y)* by *E(x)/E(y)*. A Taylor series expansion yields that to the second order *E(x/y)* >(<) *E(x)/E(y)* if and only if 1 >(<) *ρ(x,y)cv(x)/cv(y)*, where *ρ(.,.)* is the correlation coefficient and *cv(.)* the coefficient of variation. The bias will typically be positive, whereby its magnitude depends on the value of *E(x)/E(y)*.

(16) $E(\hat{Q}_C) = (1/\varsigma(S))\sum_{n=1}^{N}(q_n^1/q_n^0)\Pr(n \in S) = \sum_{n=1}^{N}(q_n^0/\sum_{n=1}^{N}q_n^0)(q_n^1/q_n^0) = Q_D$.

Put otherwise, under this sampling design, the sample Carli quantity index is an unbiased estimator of the population Dutot quantity index.

Let the total comparison period value now be unknown to the statistician and consider the sample unit value index

(17) $$\hat{P}_U \equiv \frac{\sum_{n\in S} p_n^1 q_n^1 / \sum_{n\in S} q_n^1}{\sum_{n\in S} p_n^0 q_n^0 / \sum_{n\in S} q_n^0}.$$

This presupposes that the sample is of the form $\{(p_n^t q_n^t, q_n^t); t = 0,1; n \in S\}$, that is, for every sampled element one disposes of its value and its quantity in both periods. Then one can show, in much the same way as was done in expression (14), that under simple random sampling the sample unit value index is an approximately unbiased estimator of the target unit value index P_U . Likewise, it appears that

(18) $$(\sum_{n=1}^{N} p_n^0 q_n^0)\frac{\sum_{n\in S} p_n^1 q_n^1}{\sum_{n\in S} p_n^0 q_n^0}$$

is an approximately unbiased estimator of the aggregate's total comparison period value $\sum_{n=1}^{N} p_n^1 q_n^1$. Notice that (18) has the form of a ratio estimator.

Suppose next that only sample prices are available, that is, the sample is of the form $\{p_n^0, p_n^1; n \in S\}$, and consider the sample Dutot price index, defined as

(19) $$\hat{P}_D \equiv \frac{\sum_{n\in S} p_n^1}{\sum_{n\in S} p_n^0} = \frac{(1/\varsigma(S))\sum_{n\in S} p_n^1}{(1/\varsigma(S))\sum_{n\in S} p_n^0}.$$

The second part of this expression provides the familiar interpretation of the sample Dutot price index as a ratio of unweighted average sample prices.[6] Under probability proportional to size sampling, whereby again the size of element *n* is defined as its base period quantity share, it is easily seen that, apart from a technical bias,

(20) $$E(\hat{P}_D) = E\left(\frac{\sum_{n\in S} p_n^1}{\sum_{n\in S} p_n^0}\right) \approx \frac{E\left((1/\varsigma(S))\sum_{n\in S} p_n^1\right)}{E\left((1/\varsigma(S))\sum_{n\in S} p_n^0\right)} = \frac{\sum_{n=1}^{N} p_n^1 q_n^0 / \sum_{n=1}^{N} q_n^0}{\sum_{n=1}^{N} p_n^0 q_n^0 / \sum_{n=1}^{N} q_n^0}.$$

The denominator of the right hand side ratio is the same as the denominator of the unit value index P_U . The numerators, however, differ: the target index uses comparison period quantity

[6] Clearly, taking the average of prices is the counterpart of the adding-up of quantities, *i.e.* the first makes sense if and only if the second does.

shares as weights whereas $E(\hat{P}_D)$ yields base period quantity shares as weights. Thus the sample Dutot price index will in general be a biased estimator of the unit value index. The relative bias amounts to

$$(21)\quad \frac{E(\hat{P}_D)}{P_U} \approx \frac{\sum_{n=1}^{N} p_n^1 q_n^0 / \sum_{n=1}^{N} q_n^0}{\sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} q_n^1}.$$

The relative bias of the sample Dutot price index thus consists of two components, a technical part which vanishes as the sample size gets larger and a structural part which is independent of the sample size. This structural part is given by the right hand side of expression (21). It vanishes if the (relative) quantities in the comparison period are the same as those in the base period, which is unlikely to happen in practice. The result, expressed by (20), was mentioned by Balk (1994; 139) and Diewert (2002; Section 7.4).

5. Heterogeneous aggregates and the Törnqvist price index

We now turn to the more important situation where we deal with a heterogeneous aggregate. Suppose that the Törnqvist price index P_T is decided to be the target and consider its sample analogue

$$(22)\quad \hat{P}_T \equiv \prod_{n \in S} (p_n^1 / p_n^0)^{(\hat{s}_n^0 + \hat{s}_n^1)/2},$$

where $\hat{s}_n^t \equiv p_n^t q_n^t / \sum_{n \in S} p_n^t q_n^t$ (t=0,1) is element n's sample value share. It is clear that the sample must be of the form $\{(p_n^t q_n^t, p_n^t); t = 0,1; n \in S\}$, that is, for each sample element we dispose of its value and its price in both periods. Under the assumption that each element of the population has the same probability of being included in the sample, namely $\varsigma(S)/N$, it appears that

$$(23)\quad E(\ln \hat{P}_T) = \frac{1}{2} E\left(\frac{\sum_{n \in S} p_n^0 q_n^0 \ln(p_n^1 / p_n^0)}{\sum_{n \in S} p_n^0 q_n^0} + \frac{\sum_{n \in S} p_n^1 q_n^1 \ln(p_n^1 / p_n^0)}{\sum_{n \in S} p_n^1 q_n^1} \right)$$

$$\approx \frac{1}{2} \left(\frac{E((1/\varsigma(S)) \sum_{n \in S} p_n^0 q_n^0 \ln(p_n^1 / p_n^0))}{E((1/\varsigma(S)) \sum_{n \in S} p_n^0 q_n^0)} + \frac{E((1/\varsigma(S)) \sum_{n \in S} p_n^1 q_n^1 \ln(p_n^1 / p_n^0))}{E((1/\varsigma(S)) \sum_{n \in S} p_n^1 q_n^1)} \right)$$

$$= \frac{1}{2} \left(\frac{(1/N) \sum_{n=1}^{N} p_n^0 q_n^0 \ln(p_n^1 / p_n^0)}{(1/N) \sum_{n=1}^{N} p_n^0 q_n^0} + \frac{(1/N) \sum_{n=1}^{N} p_n^1 q_n^1 \ln(p_n^1 / p_n^0)}{(1/N) \sum_{n=1}^{N} p_n^1 q_n^1} \right)$$

$$= \ln P_T.$$

This means that $\ln \hat{P}_T$ is an approximately unbiased[7] estimator of $\ln P_T$. Employing Jensen's Inequality[8], one obtains

(24) $E(\hat{P}_T) \geq P_T$,

that is, the sample Törnqvist price index has an upward bias relative to its population counterpart. However, this bias is of technical nature and will approach zero when the sample size gets larger.

The previous result critically depends on the availability of sample quantity or value information. Suppose that we cannot obtain these data and consider the sample Jevons price index[9]

(25) $\hat{P}_J \equiv \prod_{n \in S} (p_n^1 / p_n^0)^{1/\varsigma(S)}$.

Under probability proportional to size sampling, whereby the size of element n is now defined as its base period value share s_n^0, resulting in $\Pr(n \in S) = \varsigma(S) s_n^0$, it is easily seen that

(26) $E(\ln \hat{P}_J) = E\left(\frac{1}{\varsigma(S)} \sum_{n \in S} \ln(p_n^1 / p_n^0) \right) = \sum_{n=1}^{N} s_n^0 \ln(p_n^1 / p_n^0) = \ln\left(\prod_{n=1}^{N} (p_n^1 / p_n^0)^{s_n^0} \right)$.

By employing Jensen's Inequality, this leads to the result that

(27) $E(\hat{P}_J) \geq \prod_{n=1}^{N} (p_n^1 / p_n^0)^{s_n^0} \equiv P_{GL}$.

At the right hand side we have obtained the so-called Geometric Laspeyres population price index, which in general will differ from the Törnqvist population price index. The relative bias of the sample Jevons price index with respect to the Törnqvist population price index is

(28) $\frac{E(\hat{P}_J)}{P_T} \geq \prod_{n=1}^{N} (p_n^1 / p_n^0)^{(s_n^0 - s_n^1)/2}$.

The relative bias of the sample Jevons price index thus consists of two components, a technical part which vanishes as the sample size gets larger and a structural part which is independent of the sample size. This structural part is given by the right hand side of expression (28). It vanishes when base period and comparison period value shares are equal, which is unlikely to occur in practice.

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7 The bias is positive, since $cv(p_n^t q_n^t \ln(p_n^1 / p_n^0)) \leq cv(p_n^t q_n^t)$ and $\rho(p_n^t q_n^t \ln(p_n^1 / p_n^0), p_n^t q_n^t) \leq 1$ (t=0,1).

8 Jensen's Inequality says that $E(f(x)) \geq f(E(x))$ when f is a convex function of one variable and the expectation of x exists. This can be shown by expanding *f(x)* as a Taylor series around *E(x)* and taking the expectation.

9 See also Bradley (2001, 379).

Instead of defining the size of element n as its base period value share s_n^0 , one could as well define its size as being $(s_n^0 + s_n^1)/2$, the arithmetic mean of its base and comparison period value share. Then we obtain, instead of (27),

$$(29)\quad E(\hat{P}_J) \geq \prod_{n=1}^{N} (p_n^1 / p_n^0)^{(s_n^0+s_n^1)/2} \equiv P_T ,$$

which means that the sample Jevons price index is an approximately unbiased estimator of the population Törnqvist price index. The bias will now vanish when the sample size gets larger. This result was mentioned by Diewert (2002; Section 7.4).

6. Heterogeneous aggregates and the Fisher price index

Suppose that instead of the Törnqvist price index one has decided that the Fisher price index (7) should be the target. Suppose further that our sample information consists of prices and quantities. The sample analogue of the population Fisher price index is

$$(30)\quad \hat{P}_F \equiv \left(\frac{\sum_{n\in S} p_n^1 q_n^0}{\sum_{n\in S} p_n^0 q_n^0} \frac{\sum_{n\in S} p_n^1 q_n^1}{\sum_{n\in S} p_n^0 q_n^1} \right)^{1/2} = \left(\frac{(1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^0}{(1/\varsigma(S))\sum_{n\in S} p_n^0 q_n^0} \frac{(1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^1}{(1/\varsigma(S))\sum_{n\in S} p_n^0 q_n^1} \right)^{1/2} .$$

Then

$$(31)\quad \ln \hat{P}_F = \frac{1}{2}\Big[\ln((1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^0) - \ln((1/\varsigma(S))\sum_{n\in S} p_n^0 q_n^0) + \ln((1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^1)$$
$$- \ln((1/\varsigma(S))\sum_{n\in S} p_n^0 q_n^1)\Big] .$$

Applying Jensen's Inequality and assuming that the sample was drawn such that the probability that element n is included in the sample is equal to $\varsigma(S)/N$, we get

$$(32)\quad E(\ln((1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^0)) \leq \ln(E((1/\varsigma(S))\sum_{n\in S} p_n^1 q_n^0)) = \ln((1/N)\sum_{n=1}^{N} p_n^1 q_n^0)$$

and similar expressions for the other three parts of the right hand side of expression (31). We might expect that the four biases cancel[10], and hence

$$(33)\quad E(\ln \hat{P}_F) \approx \ln P_F .$$

Again applying Jensen's Inequality, we see that

$$(34)\quad E(\hat{P}_F) \geq P_F ,$$

[10] To the second order, the bias involved in (32) is $-(1/2)(cv(p_n^1 q_n^0))^2 / \varsigma(S)$. For the other three parts of (31) the biases have a similar structure.

which means that under simple random sampling the sample Fisher price index has an upward bias relative to its population counterpart. This bias, however, will approach zero when the sample size gets larger.

Suppose now that only sample prices are available, and consider the sample Carli price index,

$$(35)\quad \hat{P}_C \equiv \frac{1}{\varsigma(S)} \sum_{n \in S} p_n^1 / p_n^0 \quad .$$

Under probability proportional to size sampling, whereby the size of element n is defined as its base period value share s_n^0, we immediately see that

$$(36)\quad E(\hat{P}_C) = \sum_{n=1}^{N} s_n^0 (p_n^1 / p_n^0) = \frac{\sum_{n=1}^{N} p_n^1 q_n^0}{\sum_{n=1}^{N} p_n^0 q_n^0} = P_L .$$

Thus the expected value of the sample Carli price index appears to be equal to the population Laspeyres price index. This result was already mentioned by Balk (1994; 139); see also Diewert (2002; Section 7.4). The relative bias of the sample Carli price index with respect to the population Fisher price index appears to be

$$(37)\quad \frac{E(\hat{P}_C)}{P_F} = \left(\frac{P_L}{P_P} \right)^{1/2} ,$$

which is the square root of the ratio of the population Laspeyres price index and the population Paasche price index. Notice that this bias is of structural nature, *i.e.* will not disappear when the sample size gets larger.

Note that the population Fisher price index can be written as

$$(38)\quad P_F = \left(\sum_{n=1}^{N} s_n^0 (p_n^1 / p_n^0) \right)^{1/2} \left(\sum_{n=1}^{N} s_n^1 (p_n^1 / p_n^0)^{-1} \right)^{-1/2} .$$

We now consider whether, following a suggestion of Fisher (1922; 472; formula 101), the Carruthers-Sellwood-Ward (1980) - Dalén (1992) sample price index

$$(39)\quad \hat{P}_{CSWD} \equiv \left(\frac{1}{\varsigma(S)} \sum_{n \in S} (p_n^1 / p_n^0) \right)^{1/2} \left(\frac{1}{\varsigma(S)} \sum_{n \in S} (p_n^1 / p_n^0)^{-1} \right)^{-1/2}$$

under some sampling design might be a suitable estimator of the population Fisher price index. The CSWD sample price index is the geometric average of the sample Carli price index (35) and the sample Harmonic (or Coggeshall) price index

$$(40)\quad \hat{P}_H \equiv \left(\frac{1}{\varsigma(S)} \sum_{n \in S} (p_n^1 / p_n^0)^{-1} \right)^{-1} .$$

Thus, consider

$$(41)\quad \ln \hat{P}_{CSWD} = \frac{1}{2}\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)\right) - \frac{1}{2}\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)^{-1}\right).$$

Under probability proportional to size sampling, whereby the size of element *n* is defined as its base period value share s_n^0, and again using Jensen's Inequality, we see that

$$(42)\quad E\left(\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)\right)\right) \le \ln\left(E\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)\right)\right) = \ln\left(\sum_{n=1}^N s_n^0(p_n^1/p_n^0)\right) \equiv \ln P_L.$$

Similarly,

$$(43)$$
$$-E\left(\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)^{-1}\right)\right) \ge -\ln\left(E\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)^{-1}\right)\right) = \ln\left(\sum_{n=1}^N s_n^0(p_n^1/p_n^0)^{-1}\right)^{-1} \equiv \ln P_{HL},$$

where P_{HL} is called the population Harmonic Laspeyres price index. Combining these two inequalities, one might expect that the two biases cancel[11] and thus

$$(44)\quad E(\ln \hat{P}_{CSWD}) \approx \frac{1}{2}(\ln P_L + \ln P_{HL}) = \ln(P_L P_{HL})^{1/2},$$

or, again using Jensen's Inequality,

$$(45)\quad E(\hat{P}_{CSWD}) \ge (P_L P_{HL})^{1/2}.$$

The right hand side of this inequality clearly differs from the population Fisher price index. The relative bias of the CSWD sample price index with respect to the population Fisher price index is

$$(46)\quad \frac{E(\hat{P}_{CSWD})}{P_F} \ge \left(\frac{P_{HL}}{P_P}\right)^{1/2}.$$

Notice that the relative bias consists of two components, a technical component which vanishes as the sample size gets larger and a structural component which is independent of the sample size.

Instead of defining the size of element *n* as its base period value share s_n^0, one could as well define its size as being $(s_n^0 + s_n^1)/2$, the arithmetic mean of its base and comparison period value share. Then we obtain, instead of (42),

[11] To the second order the bias involved in (42) is $-(1/2)(cv(p_n^1/p_n^0))^2/\varsigma(S)$, and the bias involved in (43) is $+(1/2)(cv(p_n^0/p_n^1))^2/\varsigma(S)$.

$$(47)\quad E\left(\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)\right)\right)\le \ln\left(\sum_{n=1}^{N}\frac{1}{2}(s_n^0+s_n^1)(p_n^1/p_n^0)\right)\equiv \ln((P_L+P_{PAL})/2),$$

where

$$(48)\quad P_{PAL}\equiv \sum_{n=1}^{N}s_n^1(p_n^1/p_n^0)$$

is the population Palgrave price index. Similarly, instead of (43) we get

(49)

$$-E\left(\ln\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)^{-1}\right)\right)\ge \ln\left(\sum_{n=1}^{N}\frac{1}{2}(s_n^0+s_n^1)(p_n^1/p_n^0)^{-1}\right)^{-1}\equiv \ln((P_{HL}{}^{-1}+P_P{}^{-1})/2)^{-1}.$$

Combining these two inequalities, one might expect that the two biases cancel and thus

$$(50)\quad E(\ln \hat{P}_{CSWD})\approx \ln\left(\frac{P_L+P_{PAL}}{P_{HL}{}^{-1}+P_P{}^{-1}}\right)^{1/2},$$

or, using Jensen's Inequality,

$$(51)\quad E(\hat{P}_{CSWD})\ge\left(\frac{P_L+P_{PAL}}{P_{HL}{}^{-1}+P_P{}^{-1}}\right)^{1/2}=P_F\left(\frac{1+P_{PAL}/P_L}{1+P_P/P_{HL}}\right)^{1/2}.$$

Notice that P_P/P_{HL} is the temporal antithesis of P_{PAL}/P_L. We may therefore expect that numerator and denominator of the right hand side multiplicative factor will approximately cancel. Thus, under the probability proportional to size sampling design defined immediately before expression (47) , the CSWD sample price index turns out to be an approximately unbiased estimator of the population Fisher price index.

We finally consider the following modification of the CSWD sample price index:

$$(52)\quad \hat{P}_B\equiv\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1/p_n^0)\right)^{1/2}\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(q_n^1/q_n^0)\right)^{-1/2}\left(\frac{1}{\varsigma(S)}\sum_{n\in S}(p_n^1q_n^1/p_n^0q_n^0)\right)^{1/2}.$$

This is the product of a sample Carli price index, a sample Harmonic quantity index, and a sample Carli value index. It is straightforward to show, using the same reasoning as the previous paragraphs, that under probability proportional to size sampling, whereby the size of element *n* is defined as its base period value share s_n^0,

$$(53)\quad E(\ln \hat{P}_B)\approx\frac{1}{2}\left[\ln P_L-\ln Q_L+\ln\left(\frac{\sum_{n=1}^{N}p_n^1q_n^1}{\sum_{n=1}^{N}p_n^0q_n^0}\right)\right]=\ln P_F,$$

and thus

(54) $E(\hat{P}_B) \geq P_F$.

However, it is clear that the computation of $\hat{P}_B$ requires more information than the computation of $\hat{P}_{CSWD}$, namely all sample quantity relatives.

7. Heterogeneous aggregates and the Walsh price index

Suppose that the Walsh price index (9) was chosen as the target and that our sample information consists of prices and quantities. The sample analogue of the population Walsh price index is

$$(55)\quad \hat{P}_W \equiv \frac{\sum_{n\in S} p_n^1 (q_n^0 q_n^1)^{1/2}}{\sum_{n\in S} p_n^0 (q_n^0 q_n^1)^{1/2}} .$$

Suppose again that S is a simple random sample. Then we find that

$$(56)\quad E(\hat{P}_W) = E\left(\frac{(1/\varsigma(S))\sum_{n\in S} p_n^1 (q_n^0 q_n^1)^{1/2}}{(1/\varsigma(S))\sum_{n\in S} p_n^0 (q_n^0 q_n^1)^{1/2}} \right) \approx \frac{E((1/\varsigma(S))\sum_{n\in S} p_n^1 (q_n^0 q_n^1)^{1/2})}{E(1/\varsigma(S))\sum_{n\in S} p_n^0 (q_n^0 q_n^1)^{1/2})}$$

$$= \frac{(1/N)\sum_{n=1}^{N} p_n^1 (q_n^0 q_n^1)^{1/2}}{(1/N)\sum_{n=1}^{N} p_n^0 (q_n^0 q_n^1)^{1/2}} = P_W ,$$

which means that the sample Walsh price index is an approximately unbiased estimator of the population Walsh price index.

Suppose now that only sample prices are available. The population Walsh price index can be written as a quadratic mean of order 1 index,

$$(57)\quad P_W = \frac{\sum_{n=1}^{N} (s_n^0 s_n^1)^{1/2} (p_n^1 / p_n^0)^{1/2}}{\sum_{n=1}^{N} (s_n^0 s_n^1)^{1/2} (p_n^1 / p_n^0)^{-1/2}} ,$$

which suggests the following sample price index

$$(58)\quad \hat{P}_{BW} \equiv \frac{\sum_{n\in S} (p_n^1 / p_n^0)^{1/2}}{\sum_{n\in S} (p_n^1 / p_n^0)^{-1/2}} .$$

Since there is in the literature no name attached to this formula, expression (58) will be baptized as the Balk-Walsh sample price index. Under a probability proportional to size sampling design, whereby the size of element n is defined as $(s_n^0 s_n^1)^{1/2}$, the geometric mean of its base and comparison period value share, we find that

$$(59)\quad E(\hat{P}_{BW}) = E\left(\frac{(1/\varsigma(S))\sum_{n\in S}(p_n^1/p_n^0)^{1/2}}{(1/\varsigma(S))\sum_{n\in S}(p_n^1/p_n^0)^{-1/2}} \right) \approx \frac{E\left((1/\varsigma(S))\sum_{n\in S}(p_n^1/p_n^0)^{1/2}\right)}{E\left((1/\varsigma(S))\sum_{n\in S}(p_n^1/p_n^0)^{-1/2}\right)}$$

$$= \frac{(1/N)\sum_{n=1}^{N}(s_n^0 s_n^1)^{1/2}(p_n^1/p_n^0)^{1/2}}{(1/N)\sum_{n=1}^{N}(s_n^0 s_n^1)^{1/2}(p_n^1/p_n^0)^{-1/2}} = P_W .$$

Thus, under this sampling design, the Balk-Walsh sample price index appears to be an approximately unbiased estimator of the population Walsh price index. The bias will approximate zero when the sample size gets larger.

With help of expression (59) it is easy to demonstrate that, if the size of element n had been defined as its base period value share, s_n^0, the expectation of the Balk-Walsh sample price index would be unequal to the population Walsh price index.

8. Heterogeneous aggregates: micro-economic considerations on the choice of the sample price index

The previous three sections demonstrated that, when nothing but sample prices are available and the sampling design is restricted to one that uses only base period value share information, it is impossible to estimate any of the population superlative price indexes unbiasedly. Basically, we are left with a number of second-best alternatives, namely the sample Jevons (25), Carli (35), Harmonic (40), Carruthers-Sellwood-Ward-Dalén (39), and Balk-Walsh (58) price indexes. Is one of these to be preferred?

To assist in the choice, we consider the sample Generalized Mean price index, which is defined as

$$(60)\quad \hat{P}_{GM}(\sigma) \equiv \left(\frac{1}{\varsigma(S)} \sum_{n\in S}(p_n^1/p_n^0)^{1-\sigma} \right)^{1/(1-\sigma)} \qquad (\sigma \neq 1)$$

$$\equiv \prod_{n\in S}(p_n^1/p_n^0)^{1/\varsigma(S)} \qquad (\sigma = 1).$$

It is immediately seen that $\hat{P}_J = \hat{P}_{GM}(1)$, $\hat{P}_C = \hat{P}_{GM}(0)$, and $\hat{P}_H = \hat{P}_{GM}(2)$, whereas $\hat{P}_{CSWD} = [\hat{P}_{GM}(0)\hat{P}_{GM}(2)]^{1/2}$, and $\hat{P}_{BW} = [\hat{P}_{GM}(1/2)\hat{P}_{GM}(3/2)]^{1/2}$. However, since the Generalized Mean price index is a monotonous function of σ, we may conclude that $\hat{P}_{CSWD} \approx \hat{P}_{BW} \approx \hat{P}_{GM}(1)$. Thus these five sample price indexes are members of the same family.

Under probability proportional to size sampling, whereby the size of element n is defined as its base period value share s_n^0, one obtains that

$$(61)\quad E(\hat{P}_{GM}(\sigma)^{1-\sigma}) = \sum_{n=1}^{N} s_n^0 (p_n^1/p_n^0)^{1-\sigma} .$$

To apply Jensen's Inequality, we must distinguish between two cases. If $\sigma \leq 0$ we obtain

(62) $E(\hat{P}_{GM}(\sigma)) \leq \left(\sum_{n=1}^{N} s_n^0 (p_n^1 / p_n^0)^{1-\sigma} \right)^{1/(1-\sigma)} \equiv P_{LM}(\sigma)$,

whereas if $\sigma \geq 0$ we obtain

(63) $E(\hat{P}_{GM}(\sigma)) \geq \left(\sum_{n=1}^{N} s_n^0 (p_n^1 / p_n^0)^{1-\sigma} \right)^{1/(1-\sigma)} \equiv P_{LM}(\sigma) \quad (\sigma \neq 1)$

$$E(\hat{P}_{GM}(1) \geq \prod_{n=1}^{N} (p_n^1 / p_n^0)^{s_n^0} \equiv P_{LM}(1),$$

where $P_{LM}(\sigma)$ is the Lloyd-Moulton population price index. Economic theory teaches us that this index is exact for a Constant Elasticity of Substitution revenue function (for the producers' output side) or cost function (for the producers' input side or the consumer).[12] The parameter σ is thereby to be interpreted as the (average) elasticity of substitution. At their output side, producers are supposed to *maximize* revenue, which implies a non-positive elasticity of substitution. Producers at their input side and consumers, however, are supposed to *minimize* cost, which implies a non-negative elasticity of substitution.

In particular, the conclusion must be that, under the sampling design here assumed, the sample Jevons, Harmonic, CSWD, and Balk-Walsh price indexes are inadmissible for the producer output side since the expected value of each of these indexes would exhibit a positive substitution elasticity. The sample Carli price index is admissible, even unbiased, but would imply a zero substitution elasticity.

9. Heterogeneous aggregates and the Lowe price index

We now turn to the more realistic case in which the Lowe price index (11) is defined to be the target. The population Lowe price index can be written as a ratio of two Laspeyres price indexes

(64) $$P_{Lo} = \frac{\sum_{n=1}^{N} p_n^1 q_n^b / \sum_{n=1}^{N} p_n^b q_n^b}{\sum_{n=1}^{N} p_n^0 q_n^b / \sum_{n=1}^{N} p_n^b q_n^b} = \frac{\sum_{n=1}^{N} s_n^b (p_n^1 / p_n^b)}{\sum_{n=1}^{N} s_n^b (p_n^0 / p_n^b)},$$

where s_n^b is element n's value share in period b (n=1,…,N), which is assumed to be some period prior to the base period. This suggests the following sample price index[13]

(65) $$\hat{P}_{Lo} \equiv \frac{\sum_{n \in S} p_n^1 / p_n^b}{\sum_{n \in S} p_n^0 / p_n^b},$$

which is the ratio of two sample Carli price indexes. Indeed, under a probability proportional to size sampling design, whereby the size of element n is defined as s_n^b, that is its period b value share, it is easily demonstrated that

[12] For the consumer case, see Balk (2000).

[13] See also Bradley (2001, 377). Note that Bradley uses 'modified Laspeyres index' instead of 'Lowe index'.

$$(66)\quad E(\hat{P}_{Lo}) = E\left(\frac{(1/\varsigma(S))\sum_{n\in S} p_n^1 / p_n^b}{(1/\varsigma(S))\sum_{n\in S} p_n^0 / p_n^b}\right) \approx \frac{E\left((1/\varsigma(S))\sum_{n\in S} p_n^1 / p_n^b\right)}{E\left((1/\varsigma(S))\sum_{n\in S} p_n^0 / p_n^b\right)}$$

$$= \frac{(1/N)\sum_{n=1}^{N} s_n^b (p_n^1 / p_n^b)}{(1/N)\sum_{n=1}^{N} s_n^b (p_n^0 / p_n^b)} = P_{Lo}$$

The bias might be expected to be positive, since in a situation of monotone price changes it will be the case that $cv(p_n^1 / p_n^b) \approx cv(p_n^0 / p_n^b)$, whereas $\rho(p_n^1 / p_n^b, p_n^0 / p_n^b) < 1$.

Alternatively and more consistent with practice, one could consider the so-called price-updated period b value shares, defined as

$$(67)\quad s_n^{b(0)} \equiv \frac{s_n^b (p_n^0 / p_n^b)}{\sum_{n=1}^{N} s_n^b (p_n^0 / p_n^b)} = \frac{p_n^0 q_n^b}{\sum_{n=1}^{N} p_n^0 q_n^b} \quad (n=1,\ldots,N).$$

Under a probability proportional to size sampling design, whereby the size of element n is now defined as $s_n^{b(0)}$, that is its price-updated period b value share, it is immediately seen that

$$(68)\quad E(\hat{P}_C) = \sum_{n=1}^{N} s_n^{b(0)} (p_n^1 / p_n^0) = P_{Lo}\ ,$$

that is, the sample Carli price index is an unbiased estimator of the population Lowe price index. However, if the size of element n was defined as s_n^b, that is its period b value share itself, one would have obtained

$$(69)\quad E(\hat{P}_C) = \sum_{n=1}^{N} s_n^b (p_n^1 / p_n^0),$$

which, unless the prices have not changed between the periods b and 0, differs from the population Lowe price index.

10. The Time Reversal test and some numerical relations

When there is nothing but sample price information available, that is, the sample has the form $\{p_n^t; t = 0,1; n \in S\}$, then the menu of sample price indexes appears to be limited. For a *homogeneous aggregate* only the sample Dutot price index (19) is available. Note that this index, like the population unit value index, satisfies the Time Reversal test, that is, using obvious notation,

$$(70)\quad \hat{P}_D(p^1, p^0)\hat{P}_D(p^0, p^1) = 1.$$

However, as has been shown, under a not unreasonable sampling design, the sample Dutot price index is a biased estimator of the target unit value index.

For a *heterogeneous aggregate* one has, depending on the definition of the target price index, the choice between the sample Carli price index (35), the sample Jevons price index (25), the sample Harmonic price index (40), the sample CSWD price index (39), the sample Balk-Walsh price index (58), and the sample Lowe price index (65). The first three indexes are special cases of the sample Generalized Mean price index (60), respectively for σ= 0, 1, 2. Since the Generalized Mean price index is monotonously increasing in $1-\sigma$, we obtain the general result that

(71) $\hat{P}_{GM}(p^1, p^0; \sigma)\hat{P}_{GM}(p^0, p^1; \sigma) \geq 1$ for $\sigma < 1$

(72) $\hat{P}_{GM}(p^1, p^0; \sigma)\hat{P}_{GM}(p^0, p^1; \sigma) \leq 1$ for $\sigma > 1$,

which means that the GM price index fails the Time Reversal Test. In particular, the Carli price index and the Harmonic price index fail the Time Reversal test, that is,

(73) $\hat{P}_C(p^1, p^0)\hat{P}_C(p^0, p^1) \geq 1$,

and

(74) $\hat{P}_H(p^1, p^0)\hat{P}_H(p^0, p^1) \leq 1$.

The Jevons price index, as well as the CSWD price index and the Balk-Walsh price index satisfy the Time Reversal test, as one verifies immediately. As has been shown, under a not unreasonable sampling design, these three sample price indexes are (approximately) unbiased estimators of the Lloyd-Moulton population price index with $\sigma = 1$.

The sample Lowe price index also satisfies the Time Reversal Test. This index is, under a not unreasonable sampling design, an (approximately) unbiased estimator of the population Lowe price index.

We now turn to numerical relations between all these indexes. It is well known that

(75) $\hat{P}_H \leq \hat{P}_J \leq \hat{P}_C$,

and thus we might expect that

(76) $\hat{P}_{CSWD} = (\hat{P}_H \hat{P}_C)^{1/2} \approx \hat{P}_J$.

The magnitudes of the differences between the indexes depend on the variance of the price relatives p_n^1 / p_n^0. When all the price relatives are equal, the inequalities (75) turn into equalities. In fact, Dalén (1992) and Diewert (1995) showed that, to the second order, the following approximations hold:

(77) $\hat{P}_J \approx \hat{P}_C(1 - \frac{1}{2}\text{var}(\varepsilon))$

(78) $\hat{P}_H \approx \hat{P}_C(1 - \text{var}(\varepsilon))$

$$(79)\quad \hat{P}_{CSWD} \approx \hat{P}_C(1-\frac{1}{2}\mathrm{var}(\varepsilon)),$$

where $\mathrm{var}(\varepsilon) \equiv (1/\varsigma(S))\sum_{n\in S}\varepsilon_n{}^2$ and $\varepsilon_n \equiv (p_n^1/p_n^0 - \hat{P}_C)/\hat{P}_C \ \ (n \in S)$. In the same way[14] one can show that

$$(80)\quad \hat{P}_{BW} \approx \hat{P}_C(1-\frac{1}{2}\mathrm{var}(\varepsilon)).$$

Thus we may conclude that the sample Jevons price index, the sample CSWD price index, and the sample Balk-Walsh price index approximate each other to the second order. From the point of view of simplicity, the sample Jevons price index obviously gets the highest score.

To obtain some insight into the relation between the sample Lowe price index (65) and the sample Carli price index (35), we write the first as

$$(81)\quad \hat{P}_{Lo} \equiv \frac{\sum_{n\in S}(p_n^0/p_n^b)(p_n^1/p_n^0)}{\sum_{n\in S}p_n^0/p_n^b}.$$

Consider now the difference $\hat{P}_{Lo} - \hat{P}_C$. By straightforward manipulation of this expression one can show that

$$(82)\quad \hat{P}_{Lo} = \hat{P}_C(1+\mathrm{cov}(\delta,\varepsilon)),$$

where $\mathrm{cov}(\delta,\varepsilon) \equiv (1/\varsigma(S))\sum_{n\in S}\delta_n\varepsilon_n$, $\delta_n \equiv (p_n^0/p_n^b - \hat{P}_C(p^0,p^b))/\hat{P}_C(p^0,p^b)$, and $\varepsilon_n \equiv (p_n^1/p_n^0 - \hat{P}_C(p^1,p^0))/\hat{P}_C(p^1,p^0) \ \ (n\in S)$. Thus the difference between these two sample price indexes depends on the covariance of the relative price changes between the periods b and 0 and those between the periods 0 and 1. Whether this difference is positive or negative, large or small, is an empirical matter.

Since in practice the sample Dutot price index appears to be used frequently in the case of heterogeneous aggregates, it might be of some interest to discuss the relation between this index and the sample Jevons index. The first is a ratio of arithmetic average prices whereas the second is a ratio of geometric average prices. In order to see their relation, we write the Jevons index as

$$(83)\quad \ln\hat{P}_J = (1/\varsigma(S))\sum_{n\in S}\ln(p_n^1/p_n^0)$$

and the Dutot index as

[14] The method of proof is to write the ratio of $\hat{P}_{BW}$ to $\hat{P}_C$ as a function $f(\varepsilon)$ and expand this function as a Taylor series around 0. Notice thereby that $\sum_{n\in S}\varepsilon_n = 0$.

$$(84)\quad \ln \hat{P}_D = \sum_{n\in S}\left(\frac{L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)}{\sum_{n\in S} L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)}\right)\ln(p_n^1/p_n^0),$$

where $\bar{p}^t \equiv (1/\varsigma(S))\sum_{n\in S} p_n^t$ (t = 0,1) are the arithmetic average prices and *L(.,.)* denotes the logarithmic mean. This mean is, for any two positive numbers *a* and *b*, defined by

$$(85)\quad L(a,b) \equiv (a-b)/\ln(a/b) \text{ and } L(a,a) \equiv a.$$

It is a symmetric mean with the property that $(ab)^{1/2} \le L(a,b) \le (a+b)/2$, that is, it lies beween the geometric and the arithmetic mean.[15] Thus, $L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)$ can be interpreted as the mean relative price of element *n*. Then

$$(86)\quad \ln \hat{P}_D - \ln \hat{P}_J = \sum_{n\in S}\left(\frac{L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)}{\sum_{n\in S} L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)} - \frac{1}{\varsigma(S)}\right)\ln(p_n^1/p_n^0)$$

$$= \frac{1}{\varsigma(S)}\sum_{n\in S}\left(\frac{L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)}{(1/\varsigma(S))\sum_{n\in S} L(p_n^0/\bar{p}^0, p_n^1/\bar{p}^1)} - 1\right)(\ln(p_n^1/p_n^0) - \ln \hat{P}_J),$$

which means that the (sign of the) difference between the Dutot and the Jevons index depends on the (sign of the) covariance between relative prices and price relatives. Whether this difference is positive or negative, large or small, is an empirical matter.

11. Conclusion

The theoretical arguments advanced in the previous sections lead us to the following practical advice. The advice, to be practical, concerns simple random sampling (srs), sampling with probability proportional to base period quantity shares (in the case of a homogeneous aggregate), and sampling with probability proportional to base period or (price-updated) earlier period value shares (in the case of a heterogeneous aggregate) (pps). It is recognised that sampling in practice may take two stages: the sampling of respondents (outlets or firms) and of commodities. The discussion here was kept for simplicity in terms of single stage sampling. It is also recognised that purposive sampling and/or sampling with cut-off rules are often used at either stage. In such circumstances there are implicit sampling frames and selection rules and some judgement will be necessary as to which sample design outlined most closely corresponds to the method used, and the implications for choice of the sample index.

The following table presents the key results in the order of their appearance. In the first place, it is clear that respondents should be encouraged to provide timely data on comparison and base period values and prices (or quantities). Of course, in some areas this should be more feasible than others. In such cases sample indexes which mirror their population counterparts should be used and respondent-commodity pairs should be sampled using simple random sampling, since each sample index would then be an (approximately) unbiased estimator of the corresponding population one.

[15] A simple proof was provided by Lorenzen (1990).

Sample price index	Target price index	Sampling design	Expected value of sample index	Main equation
Unit value	Unit value	srs	Unit value	(17)
Dutot	Unit value	pps-q0	Biased estimate of target index	(20)
Törnqvist	Törnqvist	srs	Törnqvist	(24)
Jevons	Törnqvist	pps-s0	Geometric Laspeyres = Lloyd-Moulton (1)	(27)
Fisher	Fisher	srs	Fisher	(34)
Carli	Fisher	pps-s0	Laspeyres = Lloyd-Moulton (0)	(36)
CSWD	Fisher	pps-s0	Lloyd-Moulton (1)	(45)
Walsh	Walsh	srs	Walsh	(56)
Balk-Walsh	Walsh	pps-s0	Lloyd-Moulton (1)	
Generalized Mean (σ)	Lloyd-Moulton (σ)	pps-s0	Lloyd-Moulton (σ)	(62) – (63)
Lowe	Lowe	pps-sb	Lowe	(66)
Carli	Lowe	pps-sb(0)	Lowe	(68)

When this approach is not feasible and the best one can obtain is a sample of (matched) prices, the sampling design should be such that important elements have a correspondingly higher probability of inclusion in the sample than unimportant elements. With respect to the sample price index to be used:

- For a homogeneous aggregate, that is an aggregate for which the quantities of the elements can be meaningfully added up, one should use the sample Dutot price index. Unfortunately, this index will exhibit bias, the magnitude whereof depends on the dispersion of the elementary quantity changes between the two periods compared.

- For a heterogeneous aggregate, except at the producers' output side, one could use the sample Jevons price index. Its expected value will approximate the Geometric Laspeyres price index, which is identical to the Lloyd-Moulton price index with $\sigma = 1$.

- For a heterogeneous aggregate at the producers' output side one could use a sample Generalized Mean price index with appropriately chosen parameter $\sigma \leq 0$, the limiting case being the sample Carli price index. The expected value of such a price index will approximate a Lloyd-Moulton price index.

- When the target is a Lowe price index, the sample Lowe and Carli price indexes exhibit appropriate behaviour.

In any case the time span between the two periods compared should not become too long, for the magnitude of the bias will in general grow with the length of the time span. That is, at regular time intervals one should undertake a base period change.

There remains the practical issue as to how to decide whether an aggregate is homogenous or not. The question posed in (2) above was:

Does it make (economic) sense to add up the quantities q_n^t of the elements n=1,…,N ?

For example, if the aggregate consists of 14 inch television sets, the answer must be 'no'. Brand differences, additional facilities such as stereo, wide screens and much more account for significant variations in price. Tins of a specific brand and type of food of different sizes similarly lack homogeneity, since much of the price variation will be due to tin size. Homogeneity is lacking when the item itself varies according to identifiable price-determining characteristics. In principle the conditions of sale need to be taken into account, since an item sold by one manufacturer may command a price premium since it has better delivery, warranties or other such features. The price at initiation should be defined to have the same specified conditions of sale, but there may be elements of trust in the buyer-seller relationship that are difficult to identify. Nonetheless for practical purposes items sold by different establishments for the same product are practically treated as homogenous unless there are clearly identifiable differences in the terms and conditions surrounding the sale.

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