Vector-Based Price Index

Economic Approach

By WENDY W. XI*

This breakthrough paper derives an original vector based parameter-free and specification-free price index. This price index, to be known as the Xi price index, is simple, clean, mathematically-sound and built on price and quantity vectors. It can pass all existing reasonable price index tests, and it is not based on any utility/cost function, is consistent with all fundamental microeconomic approaches and macroeconomics observations. It consists of a Price-structure-coefficient index that measures structural inflation. Therefore the Xi price index is fundamentally and practically superior to all current price indices. (JEL C43, C14, D11)

The index number problem is an interesting and well-developed branch of economics, and the CPI is a key economic statistic in measuring inflation. This topic has drawn much attention since 1994, after ex-Federal Reserve Board Chairman Alan Greenspan told Congress that he thought the CPI overstated the true rate of consumer inflation by at least 1.1 percent (Charles L. Schultze and Christopher Mackie, 2001, 1). Therefore constructing an accurate price index that eliminates this overestimation can save the government billions of dollars every year for social security and other public programs, and provide correct information for the central banks' monetary policy.

For more than a hundred year, economists have proposed many price index formulae and derived dozens of tests that an ideal index should have passed. None of the hundreds of price indices, including the "best" three superlative price indices, Fisher, Walsh and Törnqvist (Törnqvist-Theil),[1] can pass all these tests (Erwin W. Diewert, 2001, 5).

Furthermore, none of these indices provide an exact index for a general case; they are all *exact* for some specific or flexible form of homothetic utility/cost functions, therefore they must *not* be *exact* for the real world utility/cost function or preference ordering because the real world utility/cost function is unknown.

This paper provides a much better price index, the Xi price index, than the "best" three superlative price indices. The price index is constructed by the following three steps:

i. I expressed the variables in the standard utility maximization problem by their vector form: y = $\mathbf{p.q}$ = pq cosθ, p=||$\mathbf{p}$||, q = ||$\mathbf{q}$|| and cos θ = y/pq, where $\mathbf{p}$ is the aggregate price vector and p is its norm, $\mathbf{q}$ is the aggregate quantity vector and q is its norm respectively.[2] Then from the vector-formed standard utility maximization analysis, I have proven that cos θ is decomposed into two equal parts: to increase the utility level due to quantity vector direction change (defined as the quantity structural gain that is equivalent to increasing the quantity magnitude) and to cover the cost due to price vector direction change (defined as the price structural inflation that is equivalent to increasing the price magnitude). Therefore, the price index is (p^t/p^o) $(\cos\theta^t/\cos\theta^o)^{1/2}$, and the quantity index is (q^t/q^o) $(\cos\theta^t/\cos\theta^o)^{1/2}$.

ii. I then defined common dimensions for price and quantity aggregates by the Physical-unit of Reference Year Unit Money Purchasing Power (UMPP) such that individual prices and quantities are additive with the common units (the dimensionlization does not change the price and quantity of a good). Applying these dimensionalized vectors to the above formulae results in the new price and quantity index formula.

iii. Like the Fisher price index, I then take the geometric mean of base year referenced UMPP and current year referenced UMPP dimensionlized price indices as the final Xi price index.

I calculated the Xi price indices for 11-year U.S. crops and Fruits & Nuts (1992-2003) [3] as shown in Figure 1, Figure 2 and Figure 3. The 11-year average annual Xi price index is 1.02-percentage-points less than the Laspeyres price index, 0.09-percentage-point less than the Fisher price index and 0.58-percentage-point greater than the Törnqvist index. However, the Xi price index has none of the weakness of all existing price indices; furthermore it can separate inflation in terms of the price magnitude change and the price direction change.

The Xi price index is studied in three papers. This paper constructs the Xi price and quantity indices and discusses related fundamental microeconomics concepts and topics. The second paper (Wendy Xi, 2006 a), *Test or Axiom Approach*, reveals the properties of the Xi indices by testing the Xi indices using all existing tests and discusses all existing impossible theorems. The third paper (Wendy Xi, 2006 b), *Practical, Microeconomic and Macroeconomic topics*, applies the Xi, the Fisher, the Laspeyres and the Törnqvist price indices to real world data and compares the price indices; it also shows that the Xi price index is based on and consistent with all fundamental microeconomic approaches, such as Walras' Law, General equilibrium, WARP and Stability; and it shows the instructive effects of the Xi price index on macroeconomic policy for central banks' inflation targets.

This paper is arranged as the following: Section I shows that the Cost-Of-Living index has a different dimension as a price index; defining it as a price index is due to defining real income as being constant along the same utility level. Section II uses the vector formed maximization analysis to show that along the same utility level, the real income is *not* constant, it is increasing in cosθ. Section III shows that the constant dollar difference caused by the change in cosθ can be equally decomposed into two equal parts to generate price structural inflation and quantity structural gain. Section IV defines the Xi price index, quantity index and their dimensions. Section V shows that the Xi price index is better than the Fisher price index. Section VI is a summary and concludes that from both a theoretical and practical point of view, the Xi price index should be the price index of choice for measuring the standard of living around the world.

I. Cost-Of-Living-Index Cannot Be Defined as Price Index

A. *Dilemma: Cost Index or Price Index?*

The *Cost*-Of-Living Index, COLI, is the ratio of the (minimum) *costs* of a given level of living (or *a given utility level*) in two price situations (Paul A. Samuelson and S. Swamy, 1974, 567); it is defined as the theoretical *price* index. Because the real world utility function and level is unknown, the COLI is not operational in the real world; the *Cost*-Of-Goods Index, COGI, is used instead to calculate the Consumer *Price* Index (CPI). Although COLI and COGI are conceptually different indices, the two and all existing price indices are actually *cost* indices *defined* to be *price* indices, not the real price index.

Consider the following dilemma in CPI regarding its choice of bases on either the COGI or COLI (Jack E. Triplett, 2001, 29; Schultze and Christopher Mackie, 2001, 66-68). Suppose that the price of heating is constant, but due to a very cold winter, the heating cost greatly increases. The question arises about whether we should include the cost increase in the CPI? Statistics agencies would believe the answer to be no, because the COGI based CPI is a price index and the price of heating does not change; on the other hand, a hypothetical COLI based CPI would result in an increase in the index, because to maintain a *constant standard* level of living with constant temperature, the cost increase would need to be compensated.[4]

The root reason of the dilemma is because that the cost index is defined as a price index, the wrong dimension results in a large conceptual and practical problem. Conceptually, the cost-of-living index should not be defined as a price index, because:

i. Price and cost do not have the same dimension. A price variable has a dimension of Monetary-unit/physical-unit whereas a cost variable has a monetary unit dimension only.[5]

ii. Price variable are vectors but cost variable are scalars. A scalar only has magnitude, but a vector has both magnitude and direction. The achievability of a utility level or a living standard not only depends on the magnitudes of the price and quantity, but also on the directions of the price and quantity.[6]

Therefore a cost index can be used as a price index only if cost changes directly proportional to the price change. From these discussions, we can conclude that first, the price index should not be defined as a cost-of-living index; and secondly, the price index must be vector-based to properly reflect the vector nature of price.

B. *Inconsistent Definitions on the Production Side and Consumption Side*

How COLI that has a different dimension than price index could be defined as a price index? It is because economists define that along the same utility level quantity is constant. Eeconomists all agree that a price index for the consumption side should be based on the indifference curve and a price index on the production side should be based on the PPF (Production Possibility Frontier). We will show that the definitions are inconsistent on consumption side and production side.

The aggregate price and quantity levels can be written as $y = PQ$,[7] where y is the aggregate cost, P and Q are the aggregate price and quantity levels respectively. Then, by definition, the price index and quantity index for two periods *a* and *b,* that are on the *same* utility or living standard level, can be written as

$$P^b/P^a\, Q^b/Q^a = y^b/y^a \rightarrow \mathrm{I}_P^{ba}\,\mathrm{I}_Q^{ba} = \mathrm{I}_y^{ba}$$

Economists further defined $Q^b = Q^a$ (Samuelson and Swamy, 1974, 581; Samuelson, 1983, 154; Charles Blackorby and Robert R. Russell, 1978, 235), because all "bundles" (points) lying on the same indifference curve yield the same standard of living, and thus all aggregate goods on the same utility level would be *equal*. Then we have $\mathrm{I}_Q^{ba} = Q^b/Q^a =$

1, and thus $I_P^{ba} = I_y^{ba}$, which results in a cost index being defined as a price index.

In the same way on production side, by defining real income as constant along the same PPF, we define the production side price index: the GDP deflator.[8] Let us show that the two criteria are contradictory and incorrect.

On the consumption side, along the same utility surface, the real income is defined as constant; while on the production side, along the same PPF the real output (real income) is defined as constant. Now we have two sound theoretical bases to measure real income/output – the indifference curve and the production possibility curve.

Let consider three bundles $\mathbf{q^a}$, $\mathbf{q^b}$ and $\mathbf{q^c}$ as shown in Figure 4. First look at the production side, from the production viewpoint, $\mathbf{q^b}$ and $\mathbf{q^c}$ provide the same real income/ output (Franklin. M. Fisher and Karl Shell, 1998); however from the consumption viewpoint, they cannot be regarded as providing the same real income because they do not reach the same utility level. Next look at the consumption side, from the consumption viewpoint $\mathbf{q^a}$ and $\mathbf{q^b}$ provide the same real income, however from the production viewpoint they cannot be regarded as the same real income, because with the same technology, $\mathbf{q^b}$ requires more inputs.

The criticism about the contradiction in real income definitions has nothing to do with the opposite *bias* of Laspeyres and Paasche approximations; the key is that we cannot have one definition based on a concave PPF and another based on a convex preference ordering. We all have the common knowledge that the same *subjective* satisfaction of two bundles along the same utility level does not mean the *objective* cost to produce the two bundles are the same. APPENDIX A provides a detailed discussion about real income definitions.

II. The Vector-Based Aggregate Consumer Maximization Problem

A. *The Aggregation Plane*

Now let us apply vector analysis to the microeconomic data and the aggregates of the data. First consider how price and quantity vectors are represented in a 2-goods space. Consider Figure 5, where $\mathbf{q}(\mathbf{q_1}, \ldots, \mathbf{q_m})$ and $\mathbf{p}(\mathbf{p_1}, \ldots, \mathbf{p_m})$ are the aggregate price and goods vector respectively, $m = 2$. The conventional representation of cost y is, $y = \Sigma p_i q_i = \mathbf{p.q},$ where y is the dot product of the vectors $\mathbf{p}$ and $\mathbf{q}$. The vector expression of the dot product is $\mathbf{p.q} = p\ q \cos\ \theta,$ where p and q are the norm of $\mathbf{p}$ and $\mathbf{q}$, θ is the angle between $\mathbf{p}$ and $\mathbf{q}$, and $\cos\ \theta = y/pq$. The vector $\mathbf{q}$ can be decomposed into two vector components, $\mathbf{q_t} = \mathbf{q} \sin\theta$ and $\mathbf{q_n} = \mathbf{q} \cos\ \theta$. $\mathbf{q_n}$ is the projection of $\mathbf{q}$ in the direction of the gradient of a utility function; it is also the component of $\mathbf{q}$ in the direction of vector $\mathbf{p}$.

Then consider M-dimensional goods. Let m=M, all the vector formulae and explanations stated in the last paragraph can be applied to the M-good space. We can represent the M-dimensional goods in a 2-dimensional space as shown in Figure 5.

Let us define the 2-dimensional plane as the Aggregation-plane. In the Aggregation-plane, an aggregate optimisation problem is similar to an individual optimisation problem in a 2-goods space. This is as if the aggregate solution were the solution to a microeconomic optimisation problem involving a single consumer. Appendix B states properties of the Aggregation-plane and the advantages of applying vector analysis to economics.

B. *Aggregate-Magnitude-Direction Properties*

Appendix C performs the vector-formed standard utility maximization analysis and concludes the following Aggregate-magnitude-direction properties:

i. q_n (or y/p) represents the constant dollar constrain; given a θ that determines the direction of $\mathbf{q_n}$, u is increasing in q_n. Intuitively, as shown in Figure 6, the bigger q_n is, the higher utility level can be reached; and

ii. θ represents the direction of price; given q_n, u is increasing in θ. As shown in Figure 7, the larger θ is, the higher utility level reached; and u reaches its minimum value at $\theta = 0$. At its optimal, $\cos\theta = 1$, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{q_n}$ are coincident ($\mathbf{q_n} = \mathbf{q}$).

We define θ as positive when $\mathbf{q}$ moves counter clockwise from the position in which $\mathbf{p}$ and $\mathbf{q}$ are coincident, and θ as negative otherwise.

The first property of the Aggregate-magnitude-direction is straightforward; the second property provides a proof for our conclusion in section I: along the same utility level, the constant dollar cost (y/p) is different, *and* it decreases in θ; that is along the same utility level, the smaller the θ is, the bigger constant dollar cost required to support the same living standard.

III. Cost-Structure and Aggregates Direction Change

Now let us apply the Aggregate-magnitude-direction properties to index study. Let us first consider Property i. If there is only price and quantity magnitude change, $\cos\theta^o = \cos\theta^1$, then $y^1/y^o = p^1q^1/p^oq^o = p^1/p^o\ q^1/q^o$, and we have: $y^t/y^o = (p^t/p^o)\ (q^t/q^o)$ ➔ $I_y = I_P * I_Q$, the price and quantity indices are their magnitude indices. It is much more complicated to apply property ii, than to apply property i, to price index study. This section we focus on how the aggregates direction changes affect price and quantity indices.

A. *Cost structure*

Let us define the *Cost-structure.*

DEFINITION 1: *Define Cost-structure coefficient as* $\cos\theta = y/pq = {}^s\mathbf{p}\,{}^s\mathbf{q}$, *where* ${}^s\mathbf{p}$ *and* ${}^s\mathbf{q}$ *are price and quantity direction vectors (direction cosines):* ${}^s\mathbf{p}$ $(\cos\alpha_1, \ldots \cos\alpha_i, \ldots \cos\alpha_m)$ *and* ${}^s\mathbf{q}$ $(\cos\beta_1, \ldots, \cos\beta_i, \ldots, \cos\beta_m)$, *where* $\cos\alpha_i = p_i/p$, $\sum (p_i/p)^2 = 1$ *and* $\cos\beta_i = q_i/q$, $\sum (q_i/q)^2 = 1$.

Two price and quantity situations, $(\mathbf{p^o}, \mathbf{q^o})$ and $(\mathbf{p^t}, \mathbf{q^t})$, have the same Cost-structure if and only if $\cos\theta^t = \cos\theta^o$.

B. *Quantity Magnitude Gain and Quantity Structural Gain*

There are two ways to reach a higher utility level: increasing the quantity magnitude and keep the direction constant as shown in Figure 8 bundle $\mathbf{p^a}$ and $\mathbf{p^b}$; or an inward change in quantity direction and keeping the quantity magnitude constant as shown in Figure 8 bundle $\mathbf{p^a}$ and $\mathbf{p^c}$. Let us define the utility level increased (decreased) due to the quantity magnitude change as the quantity magnitude gain (loss) and due to the quantity direction change as quantity structural gain (loss). The quantity structural loss happens when a quantity vector changes outward.

C. *Price Magnitude Inflation and Price Structural Inflation*

The quantity gain is not free. Intuitively as shown in Figure 8 when $\mathbf{p^c}$ reaches the higher utility due to direction change, θ becomes smaller, by Property ii, it moves to a more expensive cost structural situation, or we need more cost to support bundle $\mathbf{p^c}$. It is easy to understand that in this cost structural when the relatively expensive good becomes cheaper (but still relatively more expensive than the cheaper one, e.g beef vs. bread), then based on our utility maximization hypothesis, consumers would buy more of the expensive good than before, and then the cost to buy the bundle measured by constant dollar would be more expensive.

Since the cost increase is equivalent to a price increase, let us define it as price structural inflation and define the inflation due to price magnitude change as price magnitude inflation.

D. *Price-Structure-Coefficient and Quantity-Structure-Coefficient*

From above analysis, we can see that the quantity structural gain and price structural inflation (or quantity structural loss and price structural deflation) happen at the same time.

APPENDIX D conducts a quantitative study on how the aggregates directions change (or how the cost structure $\cos\theta$ changes) affect constant dollar cost, price, quantity and utility levels. It proves that the effect of the aggregate vectors direction change can be split into two equal portions, $\cos\theta = (\cos\theta)^{1/2} (\cos\theta)^{1/2}$. One $(\cos\theta)^{1/2}$ converts the direction change effects in the price to a price magnitude equivalency and the other $(\cos\theta)^{1/2}$ converts the direction change effects in the quantity to a quantity magnitude equivalency.

IV. The Vector Based Aggregates and Indices

A. *Generic Forms of Aggregates and Indices*

Let us define the following generic form Xi price level and index formulae.

DEFINITION 2: *Define the Xi Generic Form Price Level as: p $_g\eta$, where p is the price magnitude level* $p = ||\boldsymbol{p}||$ *and* $_g\eta$ *is the Price-Structure-Coefficient* $_g\eta = (\cos\theta)^{1/2}$.

DEFINITION 3**:** *Define the Xi Generic Form Indices:*

Price magnitude index: $_g\mathrm{Xi}_{pm} = p^t/p^o$.

Cost-Structure index: $_g\mathrm{Xi}_\eta = {}_g\cos\theta^t / {}_g\cos\theta^o$

Price-Structure index: $_g\mathrm{Xi}_\eta = {}_g\eta^t / {}_g\eta^o$

Price index: $_g\mathrm{Xi}_P = {}_g\mathrm{Xi}_{pm}\ {}_g\mathrm{Xi}_{p\eta} = p^t/p^o\ {}_g\eta^t/{}_g\eta^o$

Replace p with q in about formulae, you can get a set of quantity level and quantity index formulae. The indices in period t and o are:

$I_y = {}_g\mathrm{Xi}_{pm}\ {}_g\mathrm{Xi}_{p\eta}\ {}_g\mathrm{Xi}_{qm}\ {}_g\mathrm{Xi}_{q\eta} \Rightarrow y^t/y^o = (p^t/p^o)\ ({}_g\eta^t/{}_g\eta^o)(q^t/q^o)\ ({}_g\eta^t/{}_g\eta^o)$

$_g\mathrm{Xi}_{pm}\ {}_g\mathrm{Xi}_{p\eta}$ is the price index that includes price magnitude index (p^t/p^o) and price structural index (${}_g\eta^t/{}_g\eta^o$), and $_g\mathrm{Xi}_{qm}\ {}_g\mathrm{Xi}_{q\eta}$ is the quantity index that includes quantity magnitude index (q^t/q^o) and quantity structural index (${}_g\eta^t/{}_g\eta^o$).

B. *The Domain of X Indices*

First let us define the singular case:

DEFINITION 4*: Define the singular case as the situation where one aggregate vector, price (or quantity), changes its direction while another aggregate vector, quantity (or price), does not change its direction between period o and t.*

The domain of the Xi indices is Non-singular cases defined as the following:

i. $\cos\theta^o \equiv \cos\theta^t$. All individual prices and quantities change in exactly the same proportion.

ii. $\cos\theta^o = \cos\theta^t$. The aggregate price and quantity change in exactly the same proportion, but their components, the individual prices and quantities, may or may not change in the same proportions.

iii. $\cos\theta^o \neq \cos\theta^t$. The aggregate price and quantity change in different proportions.

Basically the Xi index can be applied to all the real world situations for calculating an index, because no singular case exists in the real world at the aggregate level.

C. *Common Dimensions for Aggregate Price and Quantity*

Since we cannot add one car to an apple, we need to define common dimensions for aggregating individual quantities and prices. We can derive the dimensions from a day-to-day event. Suppose that Wendy and Joe both have \$4 to buy whatever they want. Wendy bought 2 kg of apple, and Joe bought 8 liters of juice. *They both bought* \$4 *of goods and derived satisfaction from them,* regardless of the physical units of the goods, \$2/kg, or \$0.5/liter. Thus, we can simply use the Physical-unit of the Unit Money Purchasing Power Ξ, as the dimensions for individual goods and aggregate goods. For example, the price of apple measured by this dimension is \$1/per 0.5 kg since 1 dollar can buy 0.5 kg of apple. It is easy to see that the new physical unit for quantity measurement is simply the *reciprocal* of its price, $\Xi_i = 1/p_i$, so $\Xi_{\text{apple}} = 1/2 = 0.5$ kg and $\Xi_{\text{juice}} = 1/0.5 = 2$ liters.

After dimensionlization,[9] every quantity unit is equal to what one dollar can buy. If we measure every goods in terms of what one dollar can buy, then the price and quantity are *additive* with the common unit, \$1 good can be added to another \$1 good. This additive property is proved in Xi (2006 a) section III.

However the Unit Money Purchasing Power (UMPP) is still not a very clear measurement, because the UMPP is different in different years. We really cannot define a price level or quantity level for a certain economic situation without referring to another situation. Let us take the reference year's UMPP to define the new unit and dimension mathematically below.

DEFINITION 5**:** *Common quantity dimension*

Define the Physical-unit of the Reference year Unit Money Purchasing Power (UMPP), ${}_{\mathrm{r}}\Xi$*, as the common quantity dimension, where the superscript r denotes the reference year,* ${}_{r}\Xi$ *is a m-dimension diagonal matrix with positive elements* $1/p_i^r$ *on the main diagonal, and* $p^r{}_i$ *is a scalar number with the value of the norm of* $\mathbf{p}_i^r$ *in* $\mathbf{p^r}$ ($\mathbf{p}_1^r$, ..., $\mathbf{p}_m^r$) *and* $m \geq 1$.

DEFINITION 6*:* *Common price dimension*

Define the inverse matrix of ${}_{\mathrm{r}}\Xi$*,* ${}_{\mathrm{r}}\Xi^{-1}$*, as the common price dimension where* ${}_{\mathrm{r}}\Xi$ *is defined in* DEFINITION 5.

DEFINITION 7**:** *Dimensionlized quantity vector*

Define ${}_{\mathrm{r}}\mathbf{q} = {}_{\mathrm{r}}\mathbf{q}(p_1^r\mathbf{q_1}, ..., p_m^r\mathbf{q_m}) = {}_{\mathrm{r}}\Xi^{-1}\mathbf{q}$ *as the dimensionlized quantity vector at a quantity situation* $\mathbf{q} = (\mathbf{q_1}, ..., \mathbf{q_m})$ *measured by the Physical-unit of the Reference year UMPP* ${}_{\mathrm{r}}\Xi$ *(or normalized by a reference year price* $\mathbf{p^r}(\mathbf{p}_1^r, ..., \mathbf{p}_m^r)$, where ${}_{\mathrm{r}}\Xi$ and ${}_{\mathrm{r}}\Xi^{-1}$ *is defined in* DEFINITION 5 *and* DEFINITION 6 *respectively.*

DEFINITION 8: *Dimensionlized price vector*

Define ${}_{\mathrm{r}}\mathbf{p} = {}_{\mathrm{r}}\mathbf{p}\ (\mathbf{p_1}/p_1^r, ..., \mathbf{p_m}/p_m^r) = {}_{\mathrm{r}}\Xi\mathbf{p}$ *as a dimensionlized price vector at a price situation* $\mathbf{p} = (\mathbf{p_1}, ..., \mathbf{p_m})$ *measured in Common price dimension* ${}_{\mathrm{r}}\Xi^{-1}$ *(or normalized by a reference year price* $\mathbf{p^r}$ ($\mathbf{p}_1^r$, ..., $\mathbf{p}_m^r$))*, where* ${}_{\mathrm{r}}\Xi$ *and* ${}_{\mathrm{r}}\Xi^{-1}$ *is defined in* DEFINITION 5 *and* DEFINITION 6 *respectively.*[10]

Let us explain the new measurement system by the apple example.

Table 1. Measurement System Transformation

Apple		Base year (e.g. 2000)			Comparison year		
Measurement system	Transformation Coefficient λ	Price $p_i = p\lambda_i$	quantity $q_i = q\lambda_i^{-1}$	cost	price $p_i = p\lambda_i$	quantity $q_i = q\lambda_i^{-1}$	cost
\$/kg	$\lambda_1 = 1$ 1kg = 1kg	\$4.4/kg	2 kg	\$8.8	\$6.6/kg	3 kg	\$19.8
\$/lb	$\lambda_2 = 1/2.2$ 1lb = 1 kg /2.2	\$2/lb	4.4 lb	\$8.8	\$3/lb	6.6 lb	\$19.8
\$/${}_r\Xi$ r=2000	$\lambda_3 = 1/4.4$ $1\ {}_r\Xi_i = 1$ kg /4.4	\$1/${}_r\Xi_i$	8.8 ${}_r\Xi_i$	\$8.8	\$1.5/${}_r\Xi_i$	13.2 ${}_r\Xi_i$	\$19.8

* The right subscript of ${}_r\Xi_i$, i, denotes the particular good (apple) in the example.

From the Table 1, we can see the following:

REMARK 1**:** Changing from \$/kg to \$/${}_r\Xi$ is just like changing from \$/kg to \$/lb: the numerical representations of price and quantity have been changed but the actual values are unchanged; and the numerical representation of cost y is invariant to a change in the reference year, because $y^t = (p_i^t/p_i^0)\,(p_i^0 q_i^t) = p_i^t q_i^t$.

For example, in the comparison year, price is \$3/lb ≡ \$1.5/${}_r\Xi_i$ ≡ \$6.6/kg, all dollars being the comparison year dollar; and quantity is 6 lb ≡ 13.2 ${}_r\Xi_i$ ≡ 3kg; where ${}_r\Xi_i$ is the physical unit: 1 ${}_r\Xi_i$ =1kg/4.4 similar to 1 lb=1kg/2.2.

REMARK 2**:** We still have physical units in dimension Ξ_i defined as for example, pounds, tons, and kilograms. If we change the physical units from one measurement to another, for example change from pounds to tons, it will not affect the value of the aggregate and individual price and quantity.

D. *The Xi Price, Quantity Levels and Indices*

We have set the dimensions for aggregates based on ${}_r\Xi$; we can use the dimensionlized aggregates in the generic forms of the Xi generic indices defined in Section IV.A to get the Xi index formulas.[11] A question arises, which year should be the reference year in the Ξ, base year or comparison year? There are two reasons to take into consideration of both base year and comparison year. On the consumption side, consumers' tastes are not constant for base year and comparison year; and on the production side, the PPFs are not constant in base year and comparison year due to input and technology changes. If we calculate the price index based on base year's UMPP, but the monetary compensation for adjusting the change in prices (i.e. for Social Security programs) is spent on the comparison year's supply (or PPF) and preference, then the compensation may be over- or under-estimated. Therefore, like the Fisher indices, let us take the geometric means of the base year referenced and comparison year referenced Xi indices as the final practical Xi indices. Now the indices defined in Section IV.A becomes:

Xi Price Index: $\mathrm{Xi}_P = ({}_{o\Xi}\mathrm{Xi}_P\ {}_{t\Xi}\mathrm{Xi}_P)^{½} = [({}_op^t{}_o\eta^t/{}_op^o{}_o\eta^o)\ ({}_tp^t{}_t\eta^t\ /{}_tp^o{}_t\eta^o)]^{½} = {}_{ot}P^t/{}_{ot}P^o$

Xi Quantity Index: $\mathrm{Xi}_Q = ({}_{o\Xi}\mathrm{Xi}_Q\ {}_{t\Xi}\mathrm{Xi}_Q)^{½} = [({}_oq^t{}_o\eta^t/{}_oq^o{}_o\eta^o)\ ({}_tq^t{}_t\eta^t\ /{}_tq^o{}_t\eta^o)]^{1/2} = {}_{ot}Q^t/{}_{ot}Q^o$

The left subscripts ${}_{o\Xi}$ and ${}_{t\Xi}$ denote that Ξ's reference years in the indices are the base year Ξ and the comparison year Ξ respectively; the price ${}_rp^k = (\sum(p_i^k/p_i^r)^2)^{½}$, the quantity ${}_rq^k = (\sum(p_i^r q_i^k)^2)^{½}$, the price-structure-coefficient and quantity-structure-coefficient is ${}_r\eta^k = ({}_r\cos\theta^{\,k})^{½} = [y^k/({}_rp^k\,{}_rq^k)]^{1/2}$, where r denotes the reference year, r = o or t, and k denotes the periods, k = 0 or t.

The fully expanded forms of the Xi price and quantity indices are:

$$\left\{ \sqrt{\frac{\sqrt{\sum_{i=1}^{m}\left(\frac{p_i^1}{p_i^0}\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(\frac{p_i^0}{p_i^1}\right)^2}}} \right\}^{\frac{1}{2}} \left\{ \frac{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)}{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)} \Big/ \sqrt{\frac{\sqrt{\sum_{i=1}^{m}\left(p_i^0 q_i^1\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)^2}} \frac{\sqrt{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(p_i^1 q_i^0\right)^2}}} \right\}^{\frac{1}{2}}$$

and

$$\left\{ \sqrt{\frac{\sqrt{\sum_{i=1}^{m}\left(p_i^0 q_i^1\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)^2}} \frac{\sqrt{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(p_i^1 q_i^0\right)^2}}} \right\}^{\frac{1}{2}} \left\{ \frac{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)}{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)} \Big/ \sqrt{\frac{\sqrt{\sum_{i=1}^{m}\left(\frac{p_i^1}{p_i^0}\right)^2}}{\sqrt{\sum_{i=1}^{m}\left(\frac{p_i^0}{p_i^1}\right)^2}}} \right\}^{\frac{1}{2}}$$

The simplified Xi price and quantity indices are:

$$\left[\frac{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)}{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)} \right]^{\frac{1}{2}} \left[\frac{\sum_{i=1}^{m}\left(\frac{p_i^1}{p_i^0}\right)^2 \sum_{i=1}^{m}\left(p_i^0 q_i^0\right)^2 \sum_{i=1}^{m}\left(p_i^1 q_i^0\right)^2}{\sum_{i=1}^{m}\left(\frac{p_i^0}{p_i^1}\right)^2 \sum_{i=1}^{m}\left(p_i^0 q_i^1\right)^2 \sum_{i=1}^{m}\left(p_i^1 q_i^1\right)^2} \right]^{\frac{1}{8}}$$

and

$$\left[\frac{\sum_{i=1}^{m}\left(p_i^1 q_i^1\right)}{\sum_{i=1}^{m}\left(p_i^0 q_i^0\right)} \right]^{\frac{1}{2}} \left[\frac{\sum_{i=1}^{m}\left(\frac{p_i^0}{p_i^1}\right)^2 \sum_{i=1}^{m}\left(p_i^0 q_i^1\right)^2 \sum_{i=1}^{m}\left(p_i^1 q_i^1\right)^2}{\sum_{i=1}^{m}\left(\frac{p_i^1}{p_i^0}\right)^2 \sum_{i=1}^{m}\left(p_i^0 q_i^0\right)^2 \sum_{i=1}^{m}\left(p_i^1 q_i^0\right)^2} \right]^{\frac{1}{8}}$$

E. *Real Income, Constant Dollar and Real Purchasing Power*

In conventional terminology, *real income/cost*, has two equivalent meanings: *Constant dollar income* and *Real Purchasing Power*, they equivalently represent the "living standard". However, we have proven that along the same living standard level (same utility level), the real income measured by constant dollar *cannot* be the same, so we cannot use the real income to represent the two measurements: constant dollar income and living standard.

To avoid confusion, let us not use the term *real income*, instead use the terms Constant dollar income and Real Purchasing Power. The *Real Purchasing Power* still represents the "living standard", and the *Constant dollar income* represents "real money" that has no inflation and no money illusion (if price is doubled and income is also doubled, the constant dollar income would remain unchanged).

Let's look at the relations between cost index I_y and the Xi indices:

i. I_y/Xi_{pm} is the constant dollar index.

ii. $I_y/Xi_P = Xi_Q$, is the quantity index, which also can be thought of as the Real Purchasing Power index that measures the standard of living. If there is no Cost-structure change, then $Xi_{p\eta} = 1$ and the Real Purchasing Power index will be equal to the constant dollar index; if there is a Cost-structure change then the Cost-structure will cause price structural inflation or deflation. In such a case the constant dollar income cannot measure the real purchasing power; to maintain the same living standard we may need more or less of the constant dollar income.

The Xi price index is an inflation escalator that measures inflation in consumer goods and services; and the Xi quantity index is a Real Purchasing Power escalator that measures the living standard.[12]

V. Comparison of the Xi Index and the Fisher Index

The Fisher index is the best of the superlative price (quantity) index, because i. it provides an *exact* index to a flexible quadratic form cost (utility) function (Diewert 2000) while all other superlative price indices provide a second-order *approximation* to an arbitrary twice differentiable linearly homogenous cost (utility) function; and ii it can pass more tests than other superlative price indices (Diwert, 1993; Xi, 2006 a). Let us only compare the Xi price index with the Fisher price index.

A. *Compare Constant Dollar Measurement*

Let us first compare the measurement base of the Fisher price index and Xi price index, or which constant dollar measurements are used in the two indices. We know that cost y is invariance in dimensionlization, so the difference between different constant dollar measurements depends on how the p and q are measured or dimensionlized.

PROPOSITION 1: *The Laspeyres price index is measured by current year UMPP.*

To compare the constant dollar measurement of the two price indices, let us study their corresponding quantity indices, because a price index is in fact used primarily in order to arrive at estimates of the underlying quantity index, which measures the real purchasing power.

Let us first consider the Xi quantity magnitude index dimensionlized (or measured) by the current year constant dollar:

(1) $\quad {}_{t}Xi_{qm} = [\sum(p_i^t q_i^t)^2/\sum(p_i^t q_i^0)^2]^{1/2}$

then consider the Paasche quantity index (Pa_Q) generated by the Laspeyres price index (La_P):

(2) $\quad Pa_Q = I_y / La_P = (\mathbf{p^t q^t}/\mathbf{p^o q^o}) / (\mathbf{p^t q^o}/\mathbf{p^o q^o}) = \mathbf{p^t q^t}/\mathbf{p^t q^o} = \sum p_i^t q_i^t / \sum p_i^t q_i^0$

Comparing (1) and (2), we can see very clearly that both quantity indices are measured by current year UMPP as every component in (1) and (2), $p_i^t q_i^t$ and $p_i^t q_i^0$, are the current year UMPP measured individual quantities. The only difference is that ${}_t\mathrm{Xi}_Q$ uses the *vector* sums and Pa_Q uses the *scalar* sums, because ${}_t\mathrm{Xi}_Q$ is a vector-based quantity index and quantity is vector and Pa_Q is a cost-based quantity index and cost is scalar. Therefore Pa_Q's corresponding price index La_P is measured by current year UMPP too.

In the same way we can show the following propositions.

PROPOSITION 2: *The Paasche price index is measured by the base year UMPP.*

PROPOSITION 3: *The Fisher price index is measured by the geometric mean of based year UMPP and current year UMPP.*

B. *Compare Index Formulae*

We showed that the Fisher price index and the Xi price index use the same measurement base, same UMPP and same constant dollar, to measure the price and quantity indices, so the difference between the two indices is the index formulae.

Let us first consider one component of the Fisher quantity index, the Paasche quantity index, by an example. Consider two pairs of price and quantity: $\mathbf{p^o}(4, 0.25)$ $\mathbf{q^o}(2, 8)$ and $\mathbf{p^t}(1, 1)$ $\mathbf{q^t}(5, 5)$. The Lasperyres price index is $\mathbf{p^t q^o}/\mathbf{p^o q^o} = 1$, and its corresponding quantity index, the Paasche quantity index, is also equal to 1. We can see very clearly that with the same magnitude of aggregate quantity measured by current year UMPP as mentioned in Section V.A, the $\mathbf{q^t}$ reached the higher level of utility because of the direction changes in price and quantity. However the Paasche quantity index does not account for the *direction change*. With another similar numeric example, we can also show that the Laspeyres quantity index generated by the Paasche price index has the same problem.

The value of the Fisher price index is better than the two cost-based indices; however the Fisher index is also based on the *scalar sum* of the individual vectors. If we interpret the Fisher quantity index by the dimension Ξ, then the Fisher quantity index is:

(3) $\text{Fisher}_Q = \{[\sum(p_i^0 q_i^t)/\sum(p_i^0 q_i^0)][\sum(p_i^t q_i^t)/\sum(p_i^t q_i^0)]\}^{1/2}$

and the Xi quantity *magnitude* index is:

(4) $\text{Xi}_{qm} = \{[\sum(p_i^0 q_i^t)^2/\sum(p_i^0 q_i^0)^2]^{1/2}[\sum(p_i^t q_i^t)^2/\sum(p_i^t q_i^0)^2]^{1/2}\}^{1/2}$

Comparing (3) and (4), it is easy to see that the two indices have similar structure. The difference is that the Fisher quantity index uses the *scalar* sum of quantity, whereas the Xi quantity *magnitude* index uses the *vector* sum of quantity. We know that since quantity is a vector, the utility level reached does *not* depend on the *scalar* sum of the individual quantity vectors, but depends on the *vector sum* and the *direction* of the individual quantity vectors. So the Xi quantity level is *this vector sum* index *times* the quantity-structure index which accounts for the vector direction change effect.

Actually the only difference resulting in the indices is due to aggregates direction change. If there is no aggregates direction change, then all the cost-based indices generated the same results λ as the Xi price index. When there is a big aggregates direction change then the structural inflation did make a big difference between the Xi price index and the Fisher price index, 2-4 percentage-points as shown in Figure 1.

VI. Conclusions

Now we can conclude the following:

i. The Xi price index is a vector-based real price index, it reflects the vector nature of the price and goods, and it does not depend on any specific utility or cost functions, so it is a parameter-free and specification-free price index. All existing price indices are the cost-based indices, and the best of them (the superlative price indices) are only *exact* for some specific types of cost functions (Diewert and Nakamura, 1993), therefore it *cannot be exact* for the real world cost function.

ii. It can pass all reasonable theoretical tests that a real price index formula should pass, something that current price indices are unable to do (Xi, 2006 a).

iii. It is built on the microeconomic consumer maximization theory, and consistent with other fundamental microeconomic approaches, such as Walras' Law, General equilibrium, WARP and Stability (Xi, 2006 b).

iv. It can be used to calculate the two types of inflations, structural inflation and magnitude inflation. Therefore it has a very important implication in macroeconomics and in practice, especially for the monetary policy of central banks under the inflation targeting framework (Xi, 2006 b).

v. It is based on a consistent definition of constant dollar income for both consumption and production side, unlike the current situation where the Laspeyres price index is used for calculating the CPI and the Paasche price index is used for calculating the GDP deflator.

Therefore, the Xi index formulae can be regarded as the universal indices for measuring the living standard and international comparisons, regardless of different tastes, endowments, allocations and technologies around the world.

APPENDIX A

A.1. *Consumer Maximization Problem*

In the consumer maximization problem, the real income or real cost is not constant along a utility level. Facing a price situation, only one point on a particular utility surface minimizes the cost or the amount of real income required. This idea is thus contradictory to the definition that the real costs are equal along the same utility level.

A.2. *Consistent Definition of Real Income*

The definitions and criterions of the real income/output and the real price level do not necessarily have to be *unique*, but such rubrics should be *objectively and consistently defined*. The definition problems raised in Section I.B and Appendix A.1 happen because the definitions confused theoretical and practical analysis.

In *pure* theoretical economic analysis*, to eliminate money illusion* we normalized price and income, and then the consistent definition is either one of the following two.

i. Define the real income as constant in the *entire* goods space. An increase in the aggregate goods level, and thus reaching a higher utility level and PPF, can be regarded as a decrease in the aggregate price level.

ii. Define the real price level as constant in the *entire* goods space. An increase in the aggregate goods level, and thus reaching a higher utility level and PPF, can be regarded as an increase in the real income level.

In *pure* economics analysis, when we deal with the general equilibrium, we use the first definition: the real income is constant not only along all utility levels but along all PPFs, so no contradiction to Section I.B arises; whereas when we deal with the consumer maximization problem in A.1 we use the second definition: with the real price being constant and the real income differing along a utility level, there is a minimum real income to reach a given utility level.

However, none of the two definitions can be used to define the price index. If we use the second definition, then the price index is always 1; and if we use the first definition, then the cost index is always 1, which means that we would need the same cost in any price situation for any standard of living.

APPENDIX B

The vector analysis and the Aggregation plane are not new in economic studies (even if it was not formally defined previously) as all revealed preference theory topics are *graphically* studied in this plane. They have the following properties:

i. The vector expression looks *like* a polar expression because we can read a variable by its norm and angle, but the vector expression and analysis do *not* use polar coordinate. Let us look at the equation: $\sum p_i q_i = pq \cos\theta$. Conversional analysis uses the left side expression and vector analysis uses the right side expression. The vector expression just provides an *additional* view of the price and quantity in the same Cartesian coordinates, just as if you have a thermometer, its left side has a Fahrenheit scale and right side has Celsius scale. One reading can be translated to another.

ii. The reading translation is not arbitrary. The relation between the aggregates and their components is: $\cos\theta = y/pq = \Sigma\,(p_i/p\; q_i/q) = \Sigma \cos\alpha_i \cos\beta_i$ where p, q, $\cos\alpha_i$ and $\cos\beta_i$ are as defined in DEFINITION 1.

iii. The vector formed utility function is *not a specific* function. We have just replaced, in its conventional form, $u(q_1, \ldots q_m)$ by $u(q)$ and $\Sigma p_i q_i - y = 0$ by $q \cos\theta - q_n = 0$. They are on the same generic level; in both forms we do not specify the utility function, so none is more specific than the other. One is in cardinal form and deals only with individual price and quantity; the other is in vector form and deals with aggregates.

The advantages of using vectors to study the consumer maximization problem are:

i. Since price and quantity are both vectors, only the vector formed utility maximization problem is able to reveal to the nature of vectors and thus analyze the effect of the *directions* change in the vectors. The conventional way cannot do such analysis *quantitatively* and *mathematically*.

ii. It also has an advantage for *graphical* analysis. With the conventional view you can only see the direction of the price, not the scale of a price; with the vector view, we can see both direction and magnitude. We can illustrate in graphs the price vector **p** by the vector $\mathbf{q_n}$: the direction of **p** is the same as the direction as $\mathbf{q_n}$ since $\mathbf{q_n}$ is parallel to **p**; and the magnitude of **p** is inversely proportional to q_n since $p = y/q_n$.

iii. The vector form price index requires *less* constrained underlying optimization than the standard superlative index numbers. Both superlative price indices and the vector based price index need these conditions, transitivity, Desirability, Convexity and Continuity preference ordering, to *conduct mathematical analysis in formulas*. However the superlative price indices also *crucially* need some *specific* (even in flexible forms) cost functions (Diewert, 1993) to be the *exact Cost-of-Living* indices.

APPENDIX C

Consider the following standard consumer maximization problem: [13]

$$\max_{q} \; u(q)$$

$$\text{s.t. } \mathbf{p.q} - y = 0$$

First we need to find out how to transfer the budget constrain and price to a vector formed representation. We know that the magnitude of y and the magnitude of aggregate price p are not *independent*, and by basic demand theory with the lack of money illusion, we could only have one independent *magnitude* for *both* income and price; if income doubles and price doubles, then the constant dollar income should be the same. Therefore as shown in Figure 6, in the consumer maximization problem, we can represent the constant dollar income or budget constrain by y/p, the normalized cost, and represent the direction of the price by θ. Note that since $\mathbf{p.q} = pq \cos \theta$ and $y/p = q_n$, the consumer maximisation problem can now be rewritten into the following vector form:

$$\max_{q} \mathrm{u}(q)$$

$$\text{s.t. } q \cos \theta - q_n = 0$$

As shown in Figure 6 and Figure 7, the maximization problem in terms of θ and q_n is that given θ the consumer will maximize his utility by choosing q subject to the constant dollar constraint q_n. The Lagrangian for this problem is:

$$\mathrm{L} = \mathrm{u}(q) + \lambda (q_n - q \cos\theta)$$

The first order conditions are

$$\partial \mathrm{L}/\partial q = \partial \mathrm{u}/\partial q - \lambda \cos\theta = 0,$$

$$\partial \mathrm{L}/\partial \lambda = q_n - q \cos\theta = 0$$

To investigate the marginal effects of changes in the parameters q_n and θ in the constrain function, we apply the classical formulation of Lionel McKenzie, the standard envelope theorem of maximization, to our vector-based optimal problem. If we assume that v is well-defined in the neighborhood of our reference parameters θ and q_n, substituting the optimal value q^* into M (θ, q_n) = v $(q^*(\theta, q_n))$, then by the envelope theorem, the partial derivatives of the value function with respect to the parameters in the maximisation problem are:

$$\partial M/\partial\theta = \partial v/\partial\theta = \partial L/\partial\theta\,|_{q = q^*,\ \lambda=\lambda^*} = \lambda^* q^* \sin\theta,$$

$$\partial M/\partial q_n = \partial v/\partial q_n = \partial L/\partial q_n\,|_{q = q^*, \lambda=\lambda^*} = \lambda^*$$

Holding q fixed at its optimal value, u is increasing in θ and q_n, since $\partial M/\partial\theta > 0$ and $\partial M/\partial q_n > 0$. Next consider the optimal value of θ. Let $\partial M/\partial\theta = 0$, since $q \neq 0$ and $\lambda^* \neq 0$, when $\sin\theta = 0$, or $\theta = 0$, u reaches its optimal value. To determine sign of the optimum, check the second-order condition:

$$\partial^2 M/\partial\theta^2\,|_{\theta = 0} = = \lambda^* q^* \cos\theta\,|_{\theta = 0} = \lambda^* q^* > 0$$

We see that the second order condition for a minimum is satisfied, and thus $u|_{\theta=0} = u_{min}$.

All the prior equations can be used in any sub-space analysis.[14]

APPENDIX D

Consider the following two periods: ($\mathbf{p^a}$, $\mathbf{q^a}$) and ($\mathbf{p^b}$, $\mathbf{q^b}$) as shown in Figure 9. We can decompose it into two steps. In the first step, we only change the magnitudes of the aggregates and keep their directions constant, or keep θ constant at θ^a and change $\mathbf{q}$ from $\mathbf{q^a}$ to $\mathbf{q^c}$; and in the second step we only change directions of the aggregates and keep their magnitudes constant, or keep $||\mathbf{q^b}||=||\mathbf{q^c}||$ and change θ from θ^a to θ^b.

From a price index viewpoint, after the first step, we have indices:

(D1) $\quad y^c/y^a = (p^b/p^a)(q^b/q^a)\,(\cancel{\cos\theta^a}/\cancel{\cos\theta^a}) = (p^b/p^a)\,(q^b/q^a),$

where $y^c = y^b /(\cos\theta^b/\cos\theta^a)$. That is if there are only aggregate magnitudes changes, then the price and quantity indices are their magnitude indices. After the second step, we have indices:

(D2) $(y^c/y^a)(\cos\theta^b/\cos\theta^a) = (p^b/p^a)(q^b/q^a)(\cos\theta^b/\cos\theta^a)$ or

(D3) $y^b/y^a = (p^b/p^a)(q^b/q^a)(\cos\theta^b/\cos\theta^a)$

As shown in Figure 9, the constant dollar income increased from q_n^a to q_n^c is due to price and quantity magnitude change; and the constant dollar income increased from q_n^c to q_n^b is due to price and quantity direction. Let us study how the constant dollar cost contributes to quantity structural gain and price structural inflation.

First let us look at quantity structural gain. Figure 10 further decomposes the step 2 (the direction change) into two steps. The first step keeping θ constant at θ^a and moves the budget line L^c to L^d in parallel and let $\mathbf{q^d}$ be tangent to u^b (the destination utility level); and the second step keeping the utility level constant at u^b and moves $\mathbf{q}$ from $\mathbf{q^d}$ to $\mathbf{q^b}$. The quantity structural gain is equivalent to increasing quantity magnitude from q^b to q^d (note that $q^c = q^b$).

Next let us look at the price structural inflation as shown in Figure 11. Note that any points in the R= q_n^b cycle has the same constant dollar income budget constrain. Along the circle of R = q_n^b let us move the budget line from L^b to L^e in parallel to L^c and reach the higher level utility u^e at $\mathbf{q^e}$, which means if there is no cost-structure change then with the same amount of constant dollar income increased from q_n^c to q_n^b we can reach u^e instead of u^b.

We know that the constant dollar income spent from q_n^c to q_n^d generated the quantity structural gain from q^c (or q^b) to q^d, and thus the constant dollar income spent from q_n^d to q_n^e generated the price structural inflation. The price structural inflation is equivalent to increasing price magnitude from p^e to p^d (note that the price magnitude is reversely proportional to its quantity magnitude). We have the following relations:

(D4) $\cos\theta^b/\cos\theta^a = q^e/q^c = q^e/q^b = (q^e/q^d)\ (q^d/q^b)$,

where q^e/q^d is the *amplification* rate of price magnitude for converting price structural inflation to a price magnitude equivalency, and q^d/q^b is the *amplification* rate of quantity magnitude for converting quantity structural gain to a quantity magnitude equivalency. Therefore the θ effect can be decomposed into to the following two components:

(D5) $q^d/q^b = (\cos\theta^b/\cos\theta^a)^{\gamma}$ and $q^e/q^d = (\cos\theta^b/\cos\theta^a)^{1-\gamma}$

Since $q_n^b = q^b\cos\theta^b$, $q_n^c = q^c\cos\theta^a$, and $q^b = q^c$, we have

(D6) $\cos\theta^b/\cos\theta^a = q_n^b/q_n^c = q_n^e/q_n^c = (q_n^e/q_n^d)\ (q_n^d/q_n^c)$,

and since $q_n^d/q_n^c = q^d/q^c = q^d/q^b$ and $q_n^e/q_n^d = q^e/q^d$, we have:

(D7) $q_n^d/q_n^c = (\cos\theta^b/\cos\theta^a)^{\gamma}$ and $q_n^b/q_n^d = (\cos\theta^b/\cos\theta^a)^{1-\gamma}$

Let us introduce the following Axiom:

AXIOM 1. The quantity structural gain (loss) due to quantity direction change has to be *precisely* reflected in its monetary measurement.

Based on Axiom 1, the amplification rate for quantity structural gain, γ, is the same as its amplification rate for price structural inflation, 1- γ, so $\gamma = ½$.[15] Therefore the price and quantity magnitude levels in period t are equivalent to $p^b(\cos\theta^b/\cos\theta^a)^{1/2}$ and $q^b(\cos\theta^b/\cos\theta^a)^{1/2}$ respectively. Substituting them into (D3), we have:

(D8) $y^b/y^a = (p^b/p^a)\ (\cos\theta^b/\cos\theta^a)^{1/2}\ (q^b/q^a)\ (\cos\theta^b/\cos\theta^a)^{1/2}$

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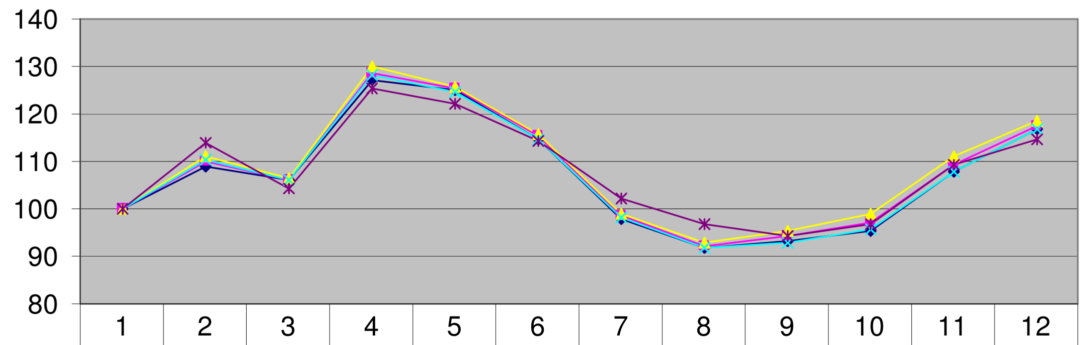

	1	2	3	4	5	6	7	8	9	10	11	12
Paasche	100	109	106.1	127	124.9	114.7	97.93	91.85	93.07	95.29	107.8	116.7
Fisher	100	110.1	106.3	128.4	125.3	115.2	98.51	92.27	94.2	97.06	109.5	117.7
Laspeyres	100	111.2	106.4	129.8	125.6	115.6	99.1	92.7	95.35	98.86	111.2	118.7
Törnqvist	100	110.2	106	128.1	124.8	114.8	98.25	91.8	92.92	95.67	107.7	116.8
XI	100	113.8	104.3	125.2	122.2	114.4	102	96.69	94.3	96.85	109.2	114.5

Figure 1. U.S. Crops and Fruits and Nuts (1992-2003) Xi Price Indices

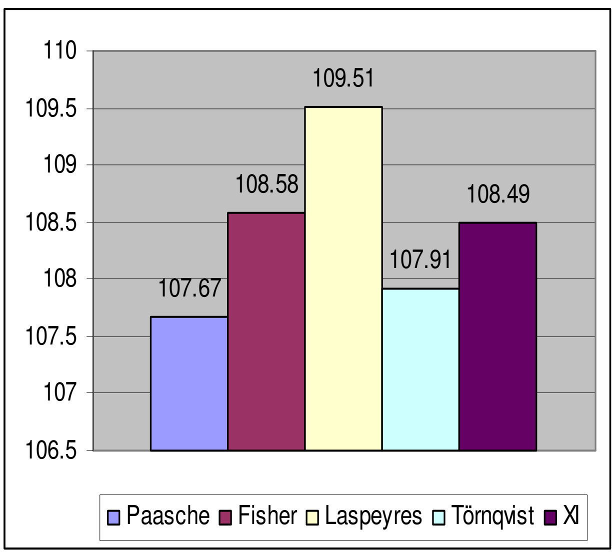

Figure 2. U.S. Crops and Fruits & Nuts (1992-2003) Average

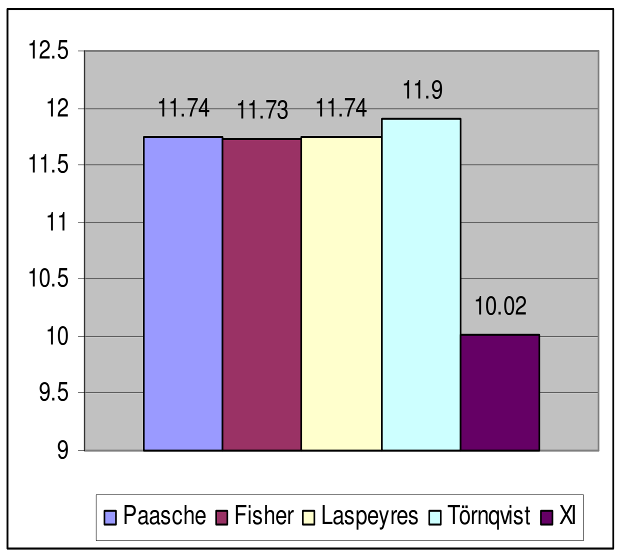

Figure 2. U.S. Crops and Fruits & Nuts (1992-2003) STD

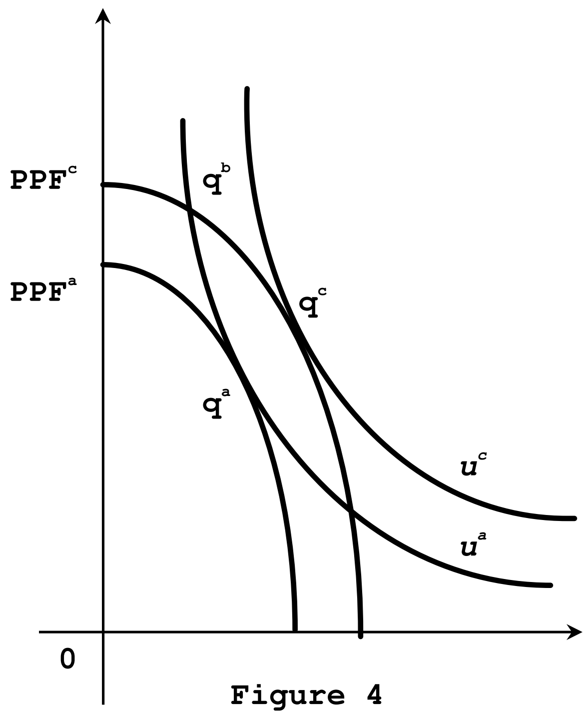

Figure 4

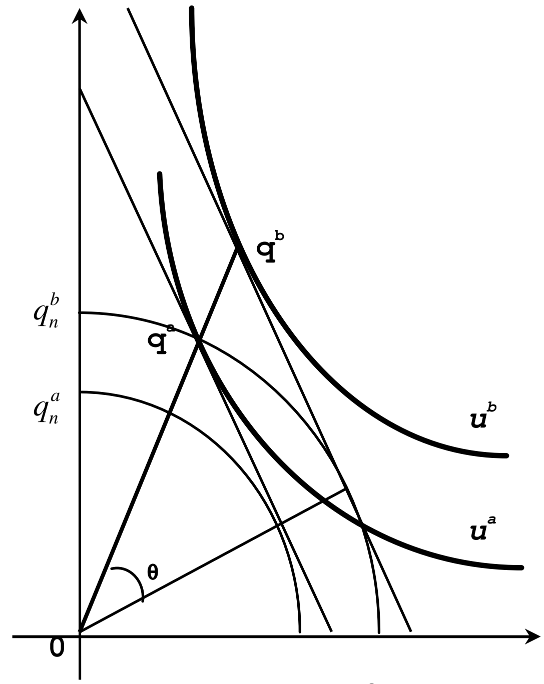

Figure 6

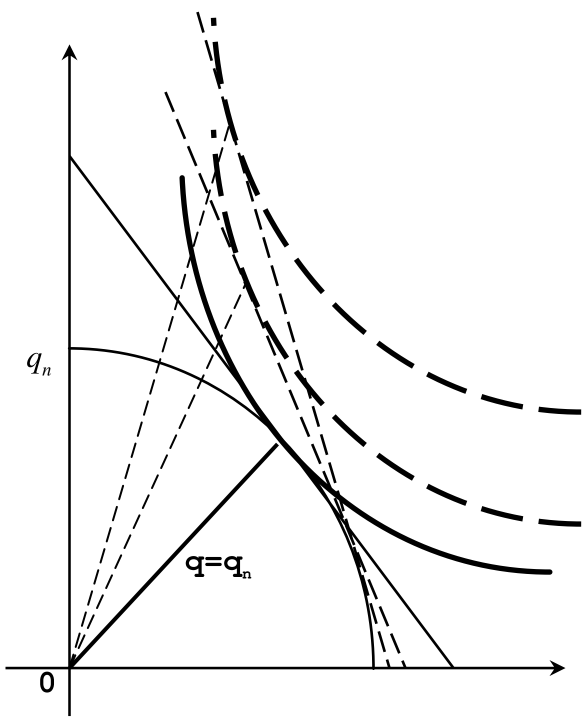

Figure 7

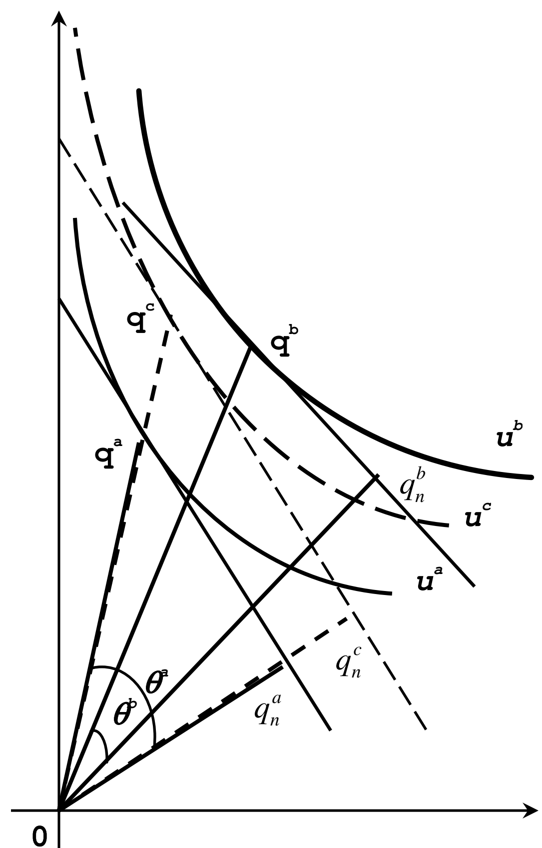

Figure 9

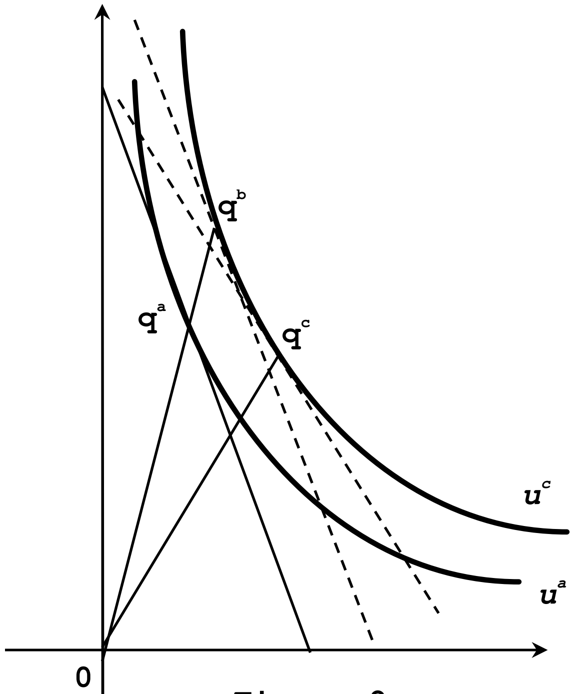

Figure 8

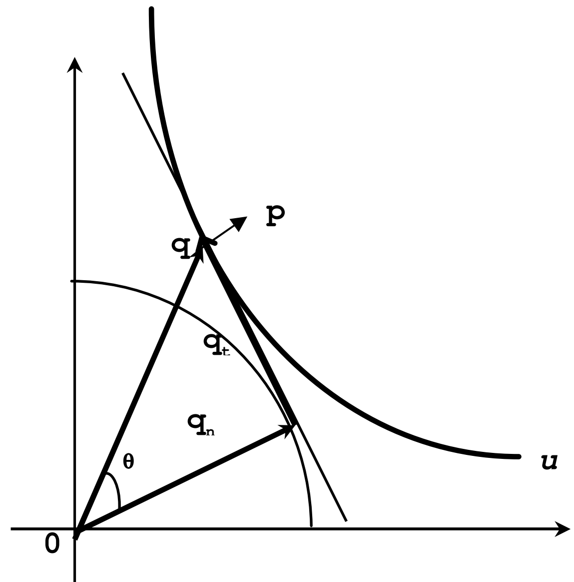

Figure 5

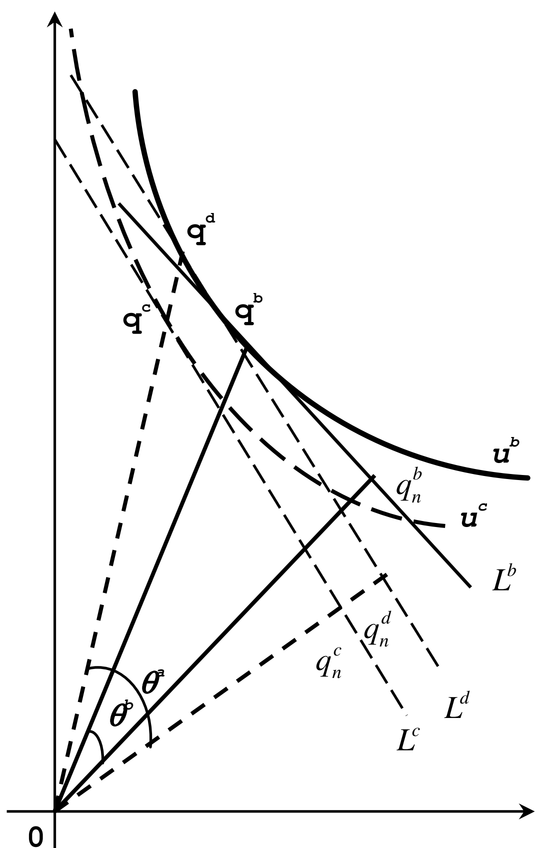

Figure 10

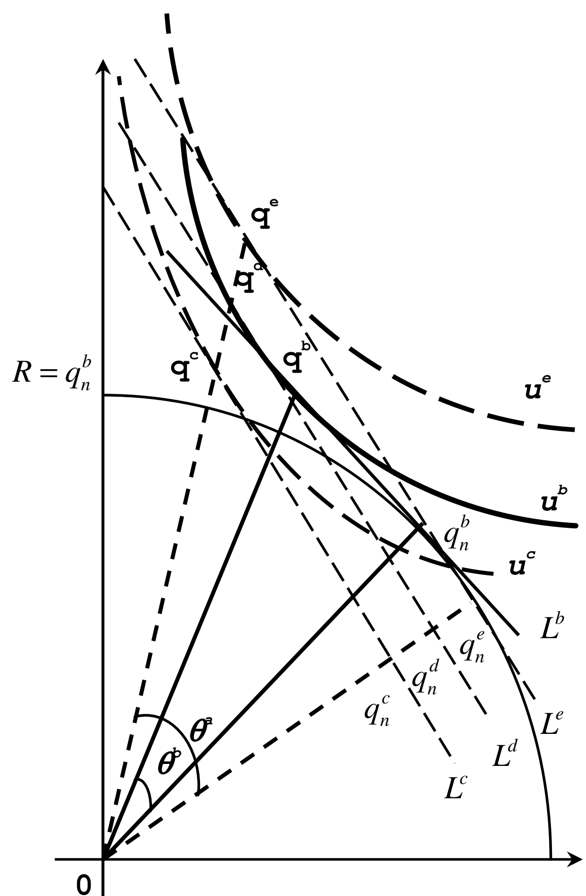

Figure 11

* Wendy Xi, Sr. Software Engineer, Global Healthcare Exchange, LLC.

2092 Grayden Court, Superior, Colorado, 80027, U.S.A.

wusanxi@gmail.com, (Work) 720-887-7193, (Home) 720 304-8144

Thanks to my son Zhou Zhou for his help, discussion and editing in making this paper possible. Thanks to Dr. Bert Balk, Dr. Ribe Martin, Professor Ben Bernanke, Professor W. Erwin Diewert and Dr. John Wood for their very helpful discussion and comments.

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[1] In August 1999, to increase accuracy, the U.S. Bureau of Labor Statistics (BLS) began to calculate the C-CPI-U (Chained Consumer Price Index for all Urban consumers) based on the Törnqvist index. The Törnqvist index is designed to reflect the substitution effect by considering the cost shares changes; therefore it is thought to be a closer approximation to the true Cost-of-Living-Index than the Laspeyres price index. The annual inflation calculated by C-CPI-U has been 0.4 percentage point less than the CPI since 2000. If the C-CPI-U were used for the past decade, then the cumulative budget deficit and federal debt would have been reduced by about $200 billion (Alan Greenspan, 2004, 5). Therefore Alan Greenspan (2004) recommended that benefits be indexed to the C-CPI, the Chained Consumer Price Index. However this price index has the common deficits mentioned above.

[2] Throughout this paper, all vectors are bolded, and their un-bolded italicized correspondences are the norm of the vectors; other scalar variables are also italicized.

[3] These are the only published raw data that can be used to re-calculate the price index using different price index formulae.

[4] Triplett (Triplett, 2001) suggested using Pollak's concept of sub-index (Robert A. Pollak, 1989) to define a CPI domain as a sub-index of COLI; thus, the CPI is an

approximation of the COLI sub-index on market-purchased goods and services, which leaves other variables such environment change, substitute effect and public goods to be dealt with by the COLI. Triplett (Triplett, 2001) thought statistics agencies then can focus on the sub-index, and central banks can consider the COLI for their targets. Such a method may clarify the domains of the two indices, but it does not help solve the root reason of the disagreement.

[5] For example, we have two rooms, room A is 1 (feet) X 19 (feet), and room B is 10(feet) X 10 (feet). The sum of sides is 40 feet for both rooms. The area index of room B to room A would be I_S= 100 feet^2/19 feet^2, not I_L = 40 feet/40 feet. One may argue that all price index formulae can be expressed in shared and price relative form, both of which are themselves dimensionless, a counterexample to the inherent contradiction, which is incorrect, because rearranging a formula can be used to interpret it from a different viewpoint, but such operation can *never* change the *nature* of the dimension. In above example, you can cancel out all the dimensions, say I_S = 100/19 and I_L=40/40, they both are dimensionless, *however we cannot say I_L is the correct index for measuring area.*

[6] As shown in Figure 8, even if $\mathbf{q^a}$ and $\mathbf{q^c}$ have the same magnitude, they reach different utility levels because they have different directions.

[7] There are a number of papers (Pollak, 1989; Samuelson and Swamy, 1974) that studies the conditions in which the cost index can be decomposed into the product of a price index and a quantity index in the context of the consumer maximization assumption. However in nature or by axiom, $y = PQ$.

[8] The theoretical product price index is analogous to COLI, even though the former is designed to create a price index that can be used to measure “real output”, and the latter

is designed to create a price index that can be used to evaluate "real income". Fisher and Shell (Fisher and Shell, 1998, 17) named the real output index as "the index of real-input usage" and the PPF as an isoquant. They stated that "With an unchanging technology, a movement along the base-period isoquant leaves the index of real-input usage unchanged", which is also incorrect as the price index definition. This is because mathematically the two definitions have no differences except that one is based on a concave curve (PPF) and the other is based on a convex curve (indifference curve); however neither of the two theoretical definitions specifies or requires the convexity-concavity of the curves.

[9] Actually, using a reference year's prices to normalize prices is quite normal in price index studies (Bert M. Balk, 1995; Wolfgang Eichhorn and Joachim Voeller, 1976; Wolfgang Eichhorn, 1978; Subramanian Swamy, 1965; R. Frisch, 1930).

[10] The dimensionlized price and quantity magnitudes have a nice interpretation from a microeconomic viewpoint. DEFINITION 8 shows that the magnitude of the dimensionlized aggregate price does not depend on the quantity, so price is an exogenous variable as assumed in the microeconomic consumer maximization problem; likewise, DEFINITION 7 shows that the magnitude of aggregate quantity depends on price, so quantity is an endogenous variable.

[11] The cost dimension is still monetary and $\cos\theta$ is a coefficient that has no dimension so both remain the same as in the Xi generic index formulae.

[12] There is a conceptual and practical gap between academic researchers and statistics agencies about the underlying framework for the price index of being COL (Cost-of-Living) or not COL (Triplett, 2001). The statistic agencies' side of economists think that

since the price index is an escalator for income and wage payments (R. Turvey, 1999), it is not a COL index, while the academic side of economists emphasizes the concept of the Cost-of-living index that considers maintaining a constant standard of living, and considers the substitute effect and other public variables (Triplett, 2001). Faulty use of the cost index as the price index was responsible for the gap. After the Xi index is introduced, there will be no such gap.

[13] In the context of price indices, Schultze and Christopher (2001, 53-57) gives a definition to a specific utility function, namely *the decision utility revealed in choice*, and assumes that consumers maximize utility by making the appropriate choices. Mas-Collell (Andreu Mas-Collell, Michael D. Whinston and Jerry R. Green, 1995, 9, 42, 46-47) concluded that if a preference relation has the properties of Completeness, Transitivity, Desirability, Convexity and Continuity then it can be represented by a utility function.

[14] In sub-space ij, $\mathbf{p_{ij}}$ ($\mathbf{p_i}$, $\mathbf{p_j}$) is the aggregate vector of $\mathbf{p_i}$ and $\mathbf{p_j}$ and $\mathbf{q_{ij}}$ ($\mathbf{q_i}$, $\mathbf{q_j}$) is the aggregate vector of $\mathbf{q_i}$ and $\mathbf{q_j}$, and θ_{ij} is the angle between $\mathbf{p_{ij}}$ and $\mathbf{q_{ij}}$, $y_{ij} = p_{ij}\, g_{ij} \cos\theta_{ij}$. $y_{ij}/p_{ij} = q_{nij}$, the indifference curve in subspace ij increases in q_{nij} and θ_{ij}, when $\theta_{ij} = 0$, $\mathbf{q_{n}}_{ij} = \mathbf{q}_{ij}$.

[15] Note that in AXIOM 1 the monetary measurement implies both constant dollar and nominal dollar measurement. We can see that when $\cos\theta^b/\cos\theta^a$ is spilt into equal portions, nominal dollar income ($y^b/y^c = \cos\theta^b/\cos\theta^a$) and constant dollar income ($q_n^b/q_n^c = \cos\theta^b/\cos\theta^a$) are all spilt into equal portions too. This axiom can be commonly accepted because monetary values mirror that of physical goods.