Aggregate Indices and Their Corresponding Elementary Indices

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11th Ottawa Group Meeting

Neuchâtel, 27-29 May 2009

*This presentation represents the author's personal opinion and does not necessarily reflect the view of the Deutsche Bundesbank or its staff.

1. Introduction and Outline of the Talk

- It is customary in official statistics for most price indices to be calculated in two stages.
- At the first stage, elementary indices are calculated on the basis of prices or their relatives, without having information on quantities or expenditures.
- At the second stage, the aggregate index is calculated on the basis of the elementary indices from the first stage, using aggregate expenditure share weights.
- “Which index formula at the elementary level, where no expenditure share weights are available, corresponds to a desired aggregate index?”

1. Introduction and Outline of the Talk

- **The existing approaches to index numbers including but not restricted to the axiomatic approach are of little guidance in choosing the elementary index corresponding to the characteristics of the index at the second stage.**

- **In order to achieve numerical equivalence between an elementary index and an arbitrary aggregate index, a statistical approach is developed.**

- **It is firstly demonstrated that every weighted index can be expressed one-to-one and onto as a “power mean” (Section 2.1).**

- **Secondly, as the solution to the problem of corresponding elementary indices depends on the joint distribution of prices and quantities, the log-normal distribution is introduced (Section 2.2).**

1. Introduction and Outline of the Talk

- **Thirdly, the log-normal distribution parameters are related to the price elasticity (Section 2.3).**

- **Finally, it is shown that the choice of the elementary indices which correspond to the desired aggregate ones can be based on the price elasticity only (Laspeyres and Paasche price indices: Section 3.1; Fisher price index: Section 3.2).**

- **This is demonstrated empirically in an application using data from German foreign trade statistics (Section 4).**

- **The conclusion gives a summary and an outlook (Section 5).**

2. Theoretical Foundations
2.1 Power Mean

- **<u>Aim</u>. It is shown that every weighted aggregate index can be written as an unweighted power mean of price relatives.**

- ***Lemma*. The price indices of Laspeyres and Paasche as well as the Fisher price index pass the Mean Value Test.**

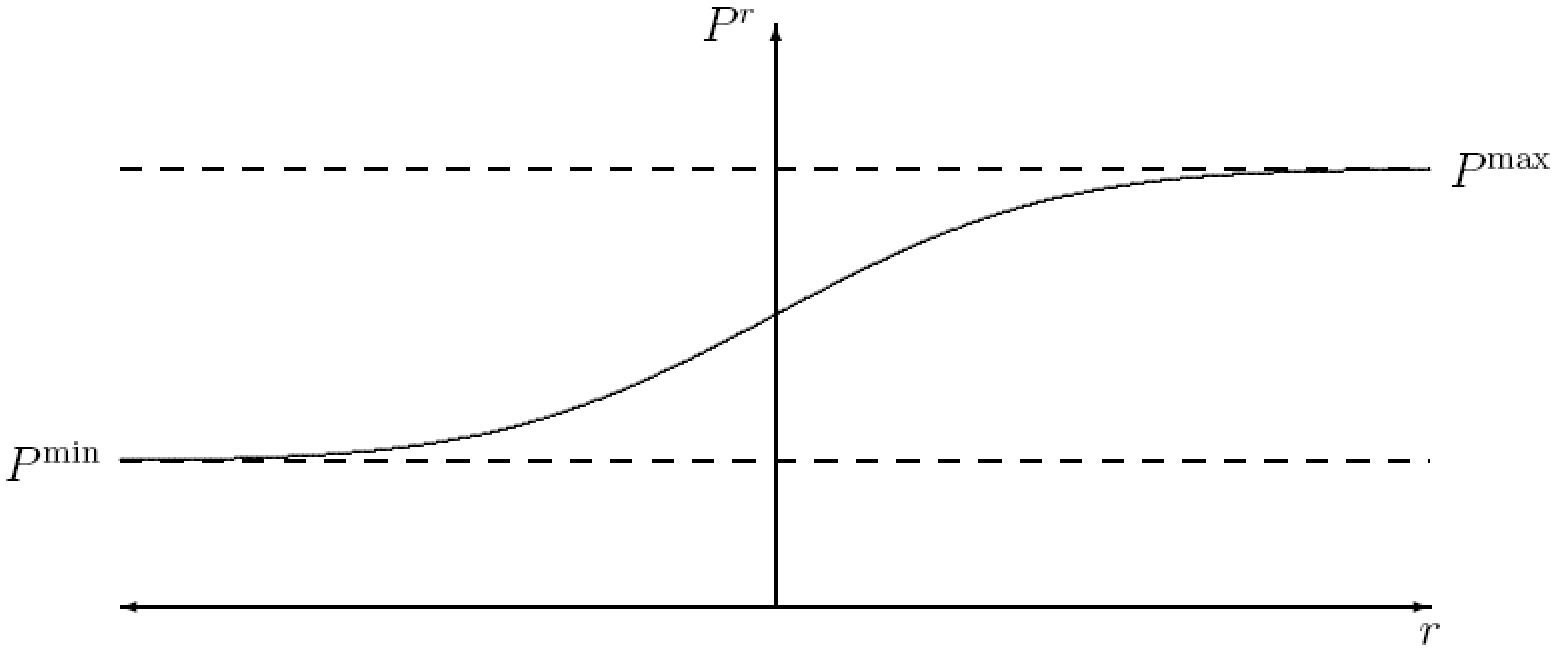
$$\min\left(\frac{p_{it}}{p_{i0}}\right) \leq P^* \leq \max\left(\frac{p_{it}}{p_{i0}}\right)$$

- ***Definition*. A “power mean” is defined as follows.**

$$P^r = \sqrt[r]{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^r}$$

2. Theoretical Foundations
2.1 Power Mean

Figure: Power Mean of Price Relatives



2. Theoretical Foundations
2.1 Power Mean

- ***Theorem*. For any aggregate index *P** that satisfies the mean value property there exists one and only one real *r* for which the power mean is numerically equivalent.**

- **By choosing the appropriate powers *r*, the resulting power means equal some of the most important elementary indices.**

2. Theoretical Foundations

2.1 Power Mean

Table: Power Means and their Formulae

r	Power Mean	Price Index	Formula
-2	reciprocal quadratic	–	$P^r(-2) = \sqrt{n / \sum_{i=1}^{n} (p_{i0}/p_{it})^2}$
-1	harmonic	Coggeshall (1887)	$P^h = n / \sum_{i=1}^{n} (p_{i0}/p_{it})$
$0^\dagger$	geometric	Jevons (1863, 1865)	$P^J = \sqrt[n]{\prod_{i=1}^{n} (p_{it}/p_{i0})}$
1	arithmetic	Carli (1764)	$P^C = \sum_{i=1}^{n} (p_{it}/p_{i0})/n$
2	quadratic	–	$P^r(2) = \sqrt{\sum_{i=1}^{n} (p_{it}/p_{i0})^2/n}$

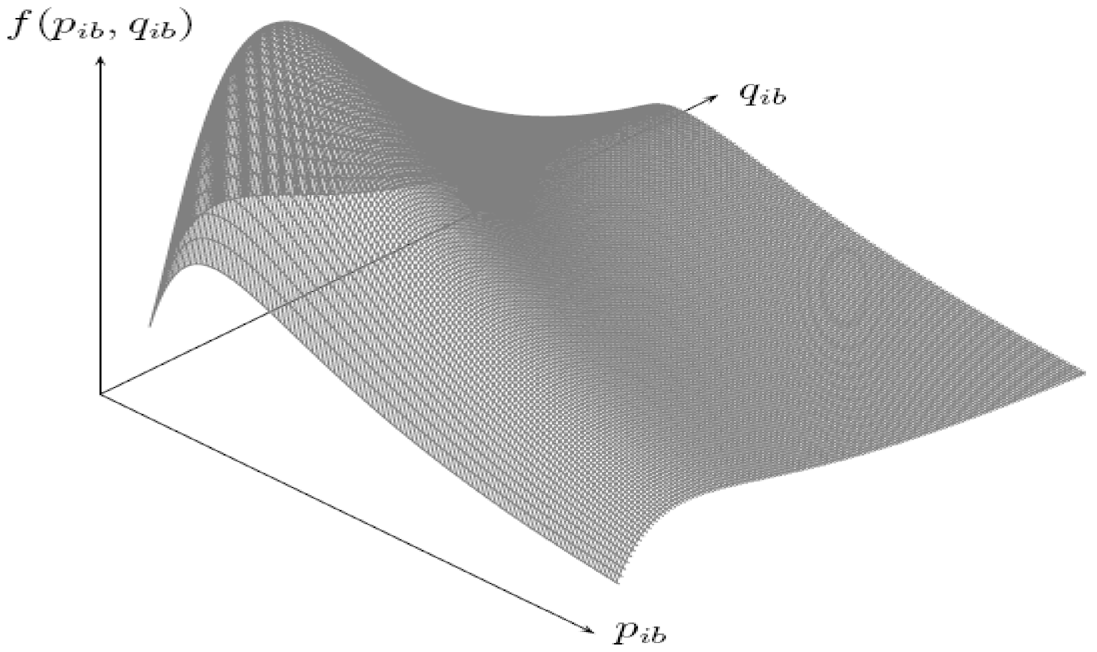
† The Jevons index is the limit of P^r as r approaches zero.

2. Theoretical Foundations
2.2 Log-Normal Distribution

- Aim. A closed form solution is provided as to which power corresponds to a given aggregate index.
- The power r cannot be derived analytically without making any further assumptions.
- *Theorem*. Under weak assumptions on the underlying data generating process, prices and quantities are jointly log-normally distributed.
- The assumption of a quadrivariate log-normal distribution of prices and quantities seems reasonable and predecessors are found in the literature.

2. Theoretical Foundations
2.2 Log-Normal Distribution

Figure: Joint Log-Normal Distribution of Prices and Quantities



2. Theoretical Foundations
2.3 Partial Adjustment Model

- **<u>Aim</u>. The implied power *r* of the Laspeyres and Paasche price indices is connected to the price elasticity derived from a partial adjustment model.**

- ***Definition*. It is assumed that there exists an equilibrium quantity traded for each good and time, and that the adjustment to this equilibrium is both incomplete and erroneous.**

$$\ln q_{ib} = (1-\rho)\alpha + \beta^* \ln p_{ib} + \rho \ln q_{ib-1} + \mathbf{x_{ib}}(1-\rho)\boldsymbol{\delta} + [(1-\rho)\eta_i^q + \varepsilon_{ib}^q]$$

- **Prices are assumed to follow an AR(1) process.**

$$\ln p_{ib} = \gamma_0 + \gamma_1 \ln p_{ib-1} + (\eta_i^p + \varepsilon_{ib}^p)$$

2. Theoretical Foundations
2.3 Partial Adjustment Model

- **Three remarks have to be made regarding the chosen model.**
- **First, the implied cross-price elasticity is zero.**
- **Second, the underlying equilibrium price elasticity β is attenuated by sluggish adjustment of quantities.**
- **Third, owing to the problem of identification with observed data on prices and quantities, the estimated effective price elasticity $\beta^* := (1 - \rho)\beta$ has to be understood as being the one of the supply-demand equilibrium rather than the one of demand.**

3. Corresponding Elementary Indices
3.1 Laspeyres & Paasche Price Indices

- **<u>Aim</u>. Combining the equations relating the power mean to the log-normal distribution parameters with those relating the log-normal distribution parameters to the model coefficients.**

- ***Theorem*. It turns out that the solution to the problem of corresponding elementary indices depends on the empirical correlation between prices and quantities. In particular, the power *r* is a function of the price elasticity only.**

$$r_L = -\beta^* \frac{1}{1-\rho\gamma_1} \approx -\beta^*$$

$$r_P \xrightarrow{t\to\infty} \beta^* \frac{1}{1-\rho\gamma_1} \approx \beta^*$$

3. Corresponding Elementary Indices
3.1 Laspeyres & Paasche Price Indices

- **From the preceding theorem, the general results for the power mean are as follows.**

- **A power mean with power equal to minus the price elasticity yields approximately the same result as the Laspeyres price index. Hence, if the price elasticity is minus one, for example, the power must equal one and the Carli index at the elementary level will correspond to the Laspeyres price index as target index.**

- **However, if the Paasche price index should be replicated, the power of the power mean must equal the price elasticity, in the above example minus one. Thus, the harmonic index gives the same result and therefore, in this case it should be used at the elementary level.**

3. Corresponding Elementary Indices
3.2 Fisher Price Index

- **Aim. Deriving the Fisher price index from the Laspeyres and Paasche price indices as their geometric mean.**

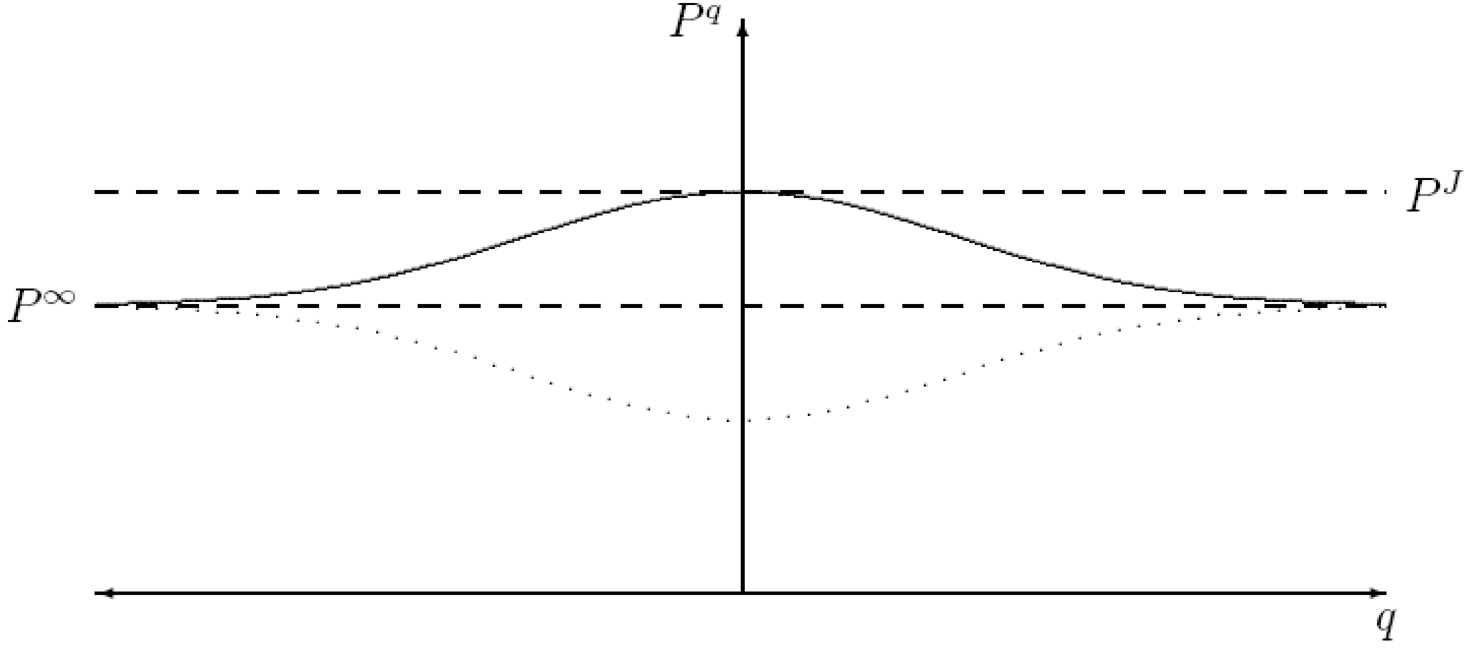
- **Owing to the symmetry of the power means which correspond to the Laspeyres and Paasche price indices, a quadratic mean corresponds to the Fisher price index.**

- ***Theorem*. A quadratic mean of order two times the absolute price elasticity corresponds to the Fisher price index.**

$$P^F \approx \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{-\beta^*}\right)^{-\frac{1}{\beta^*}}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{\beta^*}\right)^{\frac{1}{\beta^*}}} = P^q(2|\beta^*|)$$

3. Corresponding Elementary Indices
3.2 Fisher Price Index

Figure: Quadratic Mean of Price Relatives



3. Corresponding Elementary Indices
3.2 Fisher Price Index

- **This index is symmetric, i.e. $P^q = P^{-q} = P^{|q|}$.**

- **Furthermore, it is either increasing or decreasing in $|q|$, depending on the data.**

- **Note that a quadratic mean of order q, P^q, should not be mistaken for the quadratic index, $P^r(2)$.**

3. Corresponding Elementary Indices

3.2 Fisher Price Index

Table: Quadratic Means and their Formulae

q	Quadratic Mean	Formula
$0^{\dagger}$	Jevons	$P^J = \sqrt[n]{\prod_{i=1}^n (p_{it}/p_{i0})}$
1	Hybrid	$P^H = \sum_{i=1}^n \sqrt{(p_{it}/p_{i0})} / \sum_{i=1}^n \sqrt{(p_{i0}/p_{it})}$
2	CSWD	$P^{CSWD} = \sqrt{\sum_{i=1}^n (p_{it}/p_{i0})} / \sqrt{\sum_{i=1}^n (p_{i0}/p_{it})}$
3	cubic	$P^q(3) = \sqrt[3]{\sum_{i=1}^n \sqrt{(p_{it}/p_{i0})^3}} / \sqrt[3]{\sum_{i=1}^n \sqrt{(p_{i0}/p_{it})^3}}$
4	quartic	$P^q(4) = \sqrt[4]{\sum_{i=1}^n (p_{it}/p_{i0})^2} / \sqrt[4]{\sum_{i=1}^n (p_{i0}/p_{it})^2}$

† The Jevons index is the limit of P^q as q approaches zero.

4. Findings in Foreign Trade Statistics

- **<u>Aim</u>. Illustrating the methodology outlined here.**
- **An application to scanner data for homogeneous goods would be suited best.**
- **Unfortunately, scanner data are not available for the German CPI.**
- **As an empirical application, data from German foreign trade statistics are analysed.**
- **Owing to the nature of collected data, their structure is repeated cross-sections rather than a panel.**
- **Hence, estimation is performed on a pseudo panel.**

4. Findings in Foreign Trade Statistics

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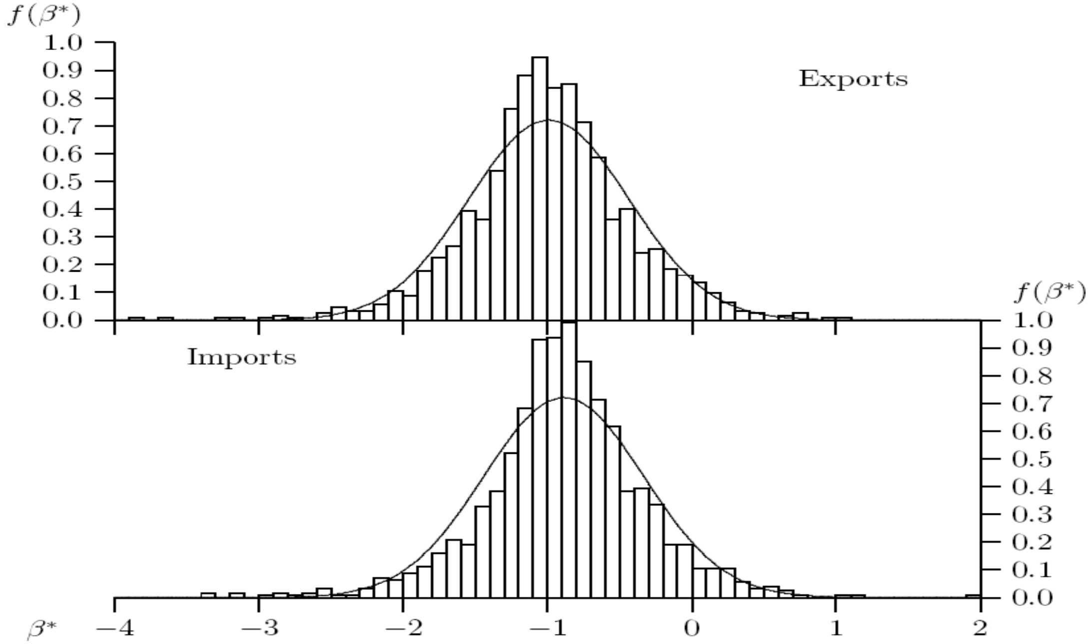
- **The data set covers 1,264 pseudo panels (nests) consisting of 12,948 groups of goods (cohorts), for exports as well as for imports, and a total of 1,839,384 observations over the period January 2000 to December 2007.**
- **The data, unit values in €1,000 per 100 kg (hereafter "prices") and weights in 100 kg (hereafter "quantities"), are transformed into their natural logarithms.**
- **Panel unit root tests show stationarity of both prices and quantities for almost all panels in exports as well as in imports.**
- **Thus, the issue of (co-)integration can safely be ignored for the remainder of the analysis.**

4. Findings in Foreign Trade Statistics

- **The price elasticity β^* is estimated in the framework of the log-linear partial adjustment model by means of dynamic panel data one-step system GMM.**
- **The histograms show positive excess kurtosis, or leptokurtosis, for exports as well as for imports.**
- **However, the distributions look both quite unimodal and symmetric.**
- **The distribution of imports lies slightly more to the right than the one of exports.**

4. Findings in Foreign Trade Statistics

Figure: Density Histogram and Normal Density Plot of β*



4. Findings in Foreign Trade Statistics

Table: Descriptive Summary Statistics of the Partial Adjustment Model

	Exports			Imports		
Statistic	β^*	$1-\rho$	Pseudo-R^2	β^*	$1-\rho$	Pseudo-R^2
Mean	−0.9911	0.8014	0.5060	−0.8877	0.8071	0.5116
Variance	0.3055	0.0466	0.0775	0.3055	0.0425	0.0748
Minimum	−3.8923	−0.0157	0.0001	−3.3727	0.0491	0.0000
Maximum	1.0826	1.6344	0.9961	1.9547	1.4127	0.9850

4. Findings in Foreign Trade Statistics

Table: Elementary Indices Corresponding to a Laspeyres Price Index

		Panels		Trade Values	
r	Price Index	Exports	Imports	Exports	Imports
0	Jevons	14%	17%	29%	28%
1	Carli	70%	72%	62%	66%
2	quadratic	15%	10%	7%	5%

- **If the Paasche price index is taken as the desired aggregate index, the corresponding power means are inverted.**

4. Findings in Foreign Trade Statistics

Table: Elementary Indices Corresponding to a Fisher Price Index

		Panels		Trade Values	
q	Price Index	Exports	Imports	Exports	Imports
0	Jevons	6%	7%	20%	17%
1	Hybrid	21%	28%	19%	25%
2	CSWD	46%	44%	48%	43%
3	cubic	21%	15%	9%	12%
4	quartic	6%	4%	3%	2%

4. Findings in Foreign Trade Statistics

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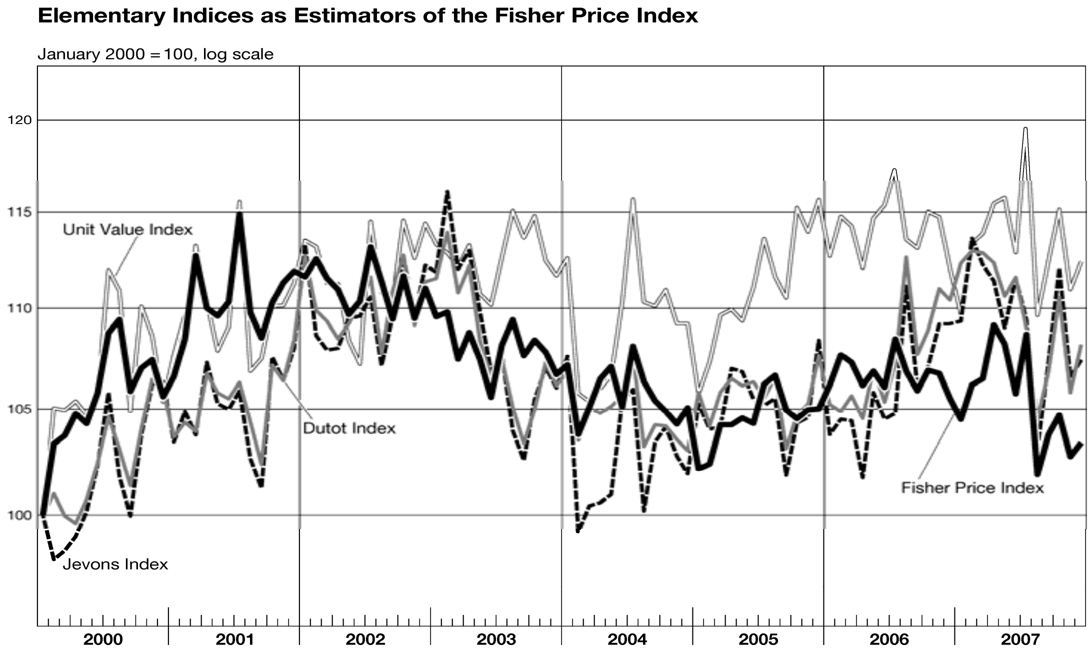
Figure: Exports of Passenger Cars



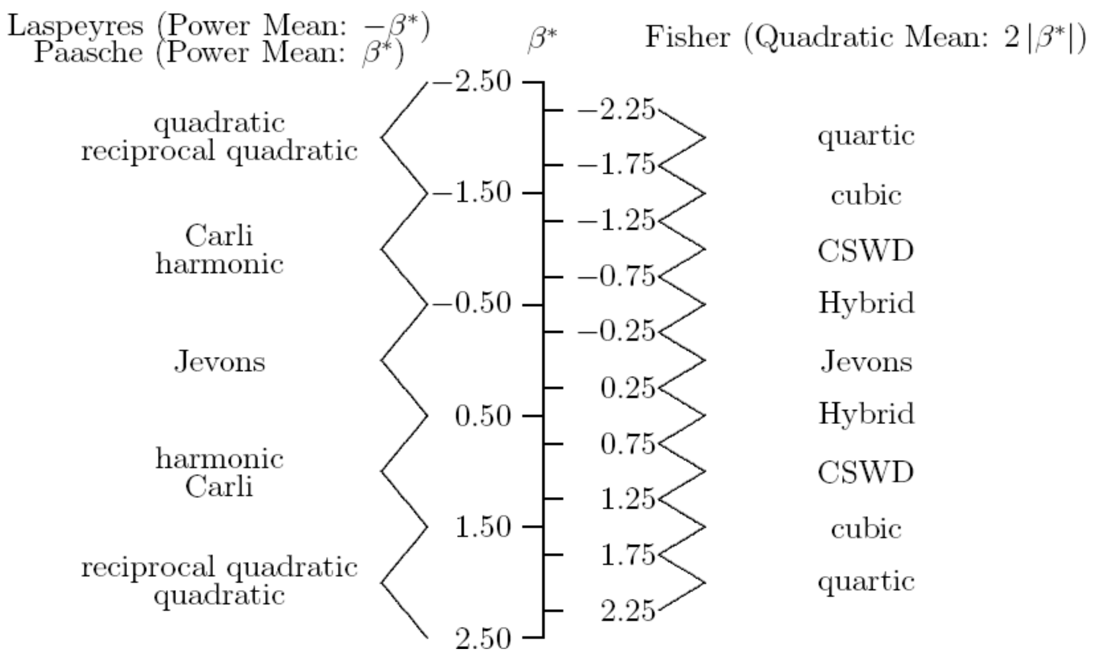
5. Conclusion

- The question of “Which index formula at the elementary level corresponds to the characteristics of the index at the aggregate level?” is answered based on “power means” and the assumption of joint log-normal distribution of prices and quantities.

- It is shown that the answer depends on the price elasticity only, which is derived from a partial adjustment model.

5. Conclusion

Figure: Overview of Corresponding Elementary Indices



5. Conclusion

- **Firstly, index calculation can be rendered more precisely if different elementary indices are applied to each group of goods, reflecting their specific price elasticities.**

- **Secondly, for different purposes, different elementary indices should be calculated. This means that if the Carli index is applied as the single formula at the elementary level of a Laspeyres price index, implying a price elasticity of minus one, for the same data, the harmonic index must be used at the elementary level of a Paasche price index.**