Using the Two-Stage Approach to Price Index Aggregation

Topic: Sampling and Elementary Aggregates; Aggregation

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Abstract

This paper assesses the practical implications for National Statistical Offices (NSOs) of implementing the two-stage approach to price index aggregation presented in the International Monetary Fund (IMF) Price Index Processor software user guide (UNECE 2009).

The two-stage approach is represented by a short term price index, which updates a long term price index. This paper examines sample change and quality adjustment within the two-stage approach.

The Australian Bureau of Statistics (ABS) is considering the implementation of the two-stage approach to price index aggregation as part of a project to update business processes, systems and methodologies used to produce price indexes.

1. Introduction

The Australian Bureau of Statistics (ABS) produces a wide range of price indexes including consumer, producer, labour, international trade and housing. Various business processes, systems and methodologies are used to compile each of these indexes.

The ABS has commenced a project to update business processes, technology and methodologies used to produce price indexes and harmonise these where possible. The ABS is considering the implementation of the Two-Stage approach to price index aggregation as part of this broader project.

This paper summarises the current approach to price index aggregation in Australia, defines the two-stage aggregation methodology and examines sample change and quality adjustment. Future challenges and likely implications for a National Statistical Office are also examined.

2. Current approach to price index production in Australia

The majority of ABS price indexes are aggregated:

a. at the elementary aggregate level using the Jevons or Dutot aggregation approaches; and

b. at the upper levels using the arithmetic Lowe or Young approaches[1].

The ABS uses a direct index approach within the elementary aggregates of most price indexes (e.g. CPI and PPI). This means that current period prices are compared to base (price reference) period prices. This can present challenges for compiling price indexes over long periods when the items being priced are replaced because they're permanently missing or no longer representative[2], and no price exists in the price reference period for new items.

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[1] Most ABS price indexes use the expenditure share form of the Lowe or Young index except for the Wage Price Index which uses the quantity form of the Laspeyres formulae at the lowest level, with quantity expressed as the number of jobs or hours.

[2] The ABS CPI is an example of this as the base period (price reference period) is updated every 6 years.

The frequency of index weight updates and sample change varies from index to index. The international trade price indexes and wage price indexes expenditure weights are updated annually, whereas the Australian consumer price index currently updates the published upper level expenditure weights every six years. Below the published level the ABS undertakes regular investigations into samples for reviews and maintenance, updating structures and weights periodically to reflect current expenditure patterns.

The ABS currently uses an index change factor to adjust direct price indexes as samples change from the previous period to the current period. The factor is calculated as the ratio of two direct indexes in the previous period, with each index representing the different samples between the previous and current period. Each factor is unique to the price index calculated, compounding from period to period.

3. Two-stage approach to aggregation

The methodology for producing price indexes by National Statistical Offices is well defined with international manuals, best practice guides and an active producer community. Countries produce a range of price indexes including consumers, producers, housing, international trade and labour costs.

Price indexes are generally produced using arithmetic Laspeyres-type indexes at the upper levels and arithmetic or geometric indexes of prices or price relatives at the elementary aggregate level. In addition, there is a range of other price index formulae that has been developed, including recent developments to compile indexes from administrative data (e.g. scanner data).

The following section outlines the derivation of the two-stage aggregation formulae from standard index formulae. Further information of the derivations can be found in Chapter 3 of the Price Index Processor Software CPI manual, IMF (2009). While the derivations are generally for the Laspeyres index, in practice the quantities used in the index generally come from a period prior to the price reference period, resulting in a Young or Lowe index being calculated. Therefore, the term "Laspeyres" or "Laspeyres-type" is used to generalise these cases. The derivation reviews an arithmetic aggregation approach[3].

[3] Attachment 1 shows the Geometric application of the two-stage approach.

4. Factorising the Laspeyres formulae

The derivation of the two-stage approach to aggregation can be shown by the factorisation of the classic Laspeyres Index representation:

$$I^t_{Laspeyres} = \frac{\sum_{i \in M_t} p_i^t q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} = \sum_{i \in M_t} \frac{p_i^0 q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} \left(\frac{p_i^t}{p_i^0} \right) \qquad (1)$$

Where M_t is the sample of items at time = t.

Further detail on notation used within the report is provided in Appendix 1.

The classical form can be expressed as the index for a component of an index, called an Elementary Aggregate (EA) and decomposed into the following factors:

$$I^t_{EA(Laspeyres)} = \sum_{i \in M_t} \frac{p_i^0 q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} \frac{p_i^t}{p_i^0} = \sum_{i \in M_t} \frac{p_i^0 q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} \frac{p_i^1}{p_i^0} \cdots \frac{p_i^t}{p_i^{t-1}} \qquad (2)$$

Let:

Expenditure share: $s_i^0 = \frac{p_i^0 q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} = \frac{w_i^0}{\sum_{i \in M_t} w_i^0}$

Long Term Price Relative (LTPR): $LTPR_i^{t-1} = \frac{p_i^1}{p_i^0} \frac{p_i^2}{p_i^1} \cdots \frac{p_i^{t-1}}{p_i^{t-2}}$

Short Term Price Relative (STPR): $STPR_i^t = \frac{p_i^t}{p_i^{t-1}}$

This leads to an intermediate, factored form:

$$I^t_{EA(Laspeyres)} = \sum_{i \in M_t} s_i^0 LTPR_i^{t-1} STPR_i^t \qquad (3)$$

Following the application of the aggregate relationship (see Appendix 2):

$$s_i^{t-1} LTPR_{EA}^{t-1} = s_i^0 LTPR_i^{t-1} \qquad (4)$$

Where, expenditure shares are price updated and rescaled with the following equation:

$$s_i^{t-1} = \frac{q_i^0 p_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} q_i^0 p_i^0 LTPR_i^{t-1}} = \frac{w_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} w_i^0 LTPR_i^{t-1}} = \frac{w_i^{t-1}}{\sum_{i \in M_t} w_i^{t-1}} \qquad (5)$$

By price updating the expenditure weights, this exactly equals the direct price index approach.

The $LTPR_{EA}^{t-1}$ is the long term price relative of the elementary aggregate, which is equivalent to the index I_{EA}^{t-1} of the price sample.

Substituting (4) into (3), the following practical and practical form of the two-stage approach to aggregation follows:

$$I_{EA(Laspeyres)}^{t} = \sum_{i \in M_t} s_i^{t-1} LTPR_{EA}^{t-1} STPR_i^t$$

$$I_{EA(Laspeyres)}^{t} = LTPR_{EA}^{t-1} \sum_{i \in M_t} s_i^{t-1} STPR_i^t$$

$$I_{EA(Laspeyres)}^{t} = LTPR_{EA}^{t-1} STPR_{EA}^{t} \qquad (6)$$

Equation 6 shows that the current period's movement from period t-1 to t using a fixed basket from period 0 can be decomposed into an aggregate Long Term Price Relative from period 0 to t-1 $\left(LTPR_{EA}^{t-1}\right)$, a price updated expenditure share $\left(s_i^{t-1}\right)$ and current period short term price relative movement $\left(STPR_i^t\right)$.

In order to incorporate the relinking of price indexes at particular points in time – incorporating the link period price index (I_{EA}^{0}), the final form of the two-stage approach to aggregation formulae is outlined by the definition described in table 1:

Table 1: Two-stage approach to aggregation: (Arithmetic formulae)

$$I_{EA(Laspeyres)}^{t} = I_{EA}^{0} LTPR_{EA}^{t-1} STPR_{EA}^{t}$$

$$STPR_{EA}^{t} = \sum_{i \in M_t} s_i^{t-1} STPR_i^t$$

Where, $s_i^{t-1} = \dfrac{w_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} w_i^0 LTPR_i^{t-1}}$; $\quad LTPR_{EA}^{t-1} = LTPR_{EA}^{t-2} STPR_{EA}^{t-1}$; $\quad LTPR_i^0 = 1$; $\quad STPR_i^t = \dfrac{p_i^t}{p_i^{t-1}}$

For all $i \in M_t$; $t = 1, 2, 3, \ldots$

5. Implementation of the two-stage approach to aggregation

The implementation of the two-stage approach to aggregation uses a two period window[4]. By using the aggregate relationship - the price updated expenditure weights relationship described in equation (4) - the aggregation procedure is able to focus purely on a two period window. One advantage of this method is there is no need to alter or use base period values when sample changes occur. In turn, the resulting index is transitive using sample change rules discussed in section 6 to retain the fixed basket approach.

6. Sample change under the two-stage aggregation approach

The current period's movement can be expressed in a form which only requires data from the previous and current periods (as shown in equation 6). This simplifies the handling of sample change each period. However, in order to maintain the fixed basket concept and not introduce bias in the index, rules surrounding the changing of sampled items must be adhered to. A formal proof of sample change impacts within a Laspeyres two-stage index are provided in appendix 3 & 4.

6.1 Adding new items

For the application of sample change for a Laspeyres Index (of the Lowe and Young forms), the addition of items to a price sample sometime after the link period must be inserted in a manner which is *equivalent* to imputing all previous period prices off the price sample movement back until the link period (t=0).

The rules that govern this process are as follows:

If at time=b, a new item j is added to the price sample, where $0 < b < t$,

In order to add an item to the sample, the following must be known:

i. Previous period price: p_j^{b-1}

ii. Current period price: p_j^b

iii. The quantity or expenditure weight of item j at a point in time, where:

   - If quantity q_j^x is known, for any time period x, the key assumption of a Laspeyres index is that quantities have not changed between the link period and period x, thus $q_j^0 = q_j^x$.

[4] This paper focuses on the Laspeyres type (arithmetic) approach to two-stage aggregation. However, see attachment 2 for the mapping of classic index formulae (both arithmetic and geometric) to the two-stage approach to aggregation.

- *For arithmetic applications:* If item weight w_j^x is known, for any time period x, the link period item weight will be derived by: $w_j^0 = \frac{w_j^x}{LTPR_{EA}^x}$, where $LTPR_{EA}^x$ is the elementary aggregate long term price relative – equivalent to the index: $I_{EA}^{0\rightarrow x}$.The above assumption regarding quantity terms is also applied here.
- If expenditure share s_j^x is known in relation to the other sampled products, for any time period x, the period x weight of item j can be found by: $w_j^x = s_j^x \sum_{i\in M_x} w_i^x$; and then apply the same procedure listed above for expenditure weights.

The application of these rules when a new item is inserted into a sample will mimic the effect of reverse imputing a new price observation off the price sample movement and re-calculating the price index.

6.2 Removing items from the basket

The removal of items is simply the process of rescaling the expenditure share element so they sum to 1. This occurs within the aggregation formulae itself, as the expenditure weights are scaled within the price sample M_t, such that the following expenditure share constraint is upheld each period: $\sum_{i\in M_t} s_i^t = 1$

6.3 Sample change with two-stage approach to aggregation – In practice

The following two tables illustrate the mechanisms present within the two-stage approach to aggregation process. The example uses the standard weighted Laspeyres Index, with each link period expenditure weight represented as: $w_i^0 = p_i^0 q_i^0$

Table 2: Example of an Elementary Aggregate price sample in the link period

i	*Expenditure Weight:* w_i^0	*Expenditure Share:* s_i^0	*Price:*		$STPR_i^1$
			Time = 0	Time = 1	
1	$w_1^0 = 1$	$s_1^0 = 1/(1+1) = 1/2$	$p_1^0 = \$1$	$p_1^1 = \$2$	$STPR_1^1 = 2/1$
2	$w_2^0 = 1$	$s_2^0 = 1/(1+1) = 1/2$	$p_2^0 = \$1$	$p_2^1 = \$1$	$STPR_2^1 = 1/1$
$STPR_{EA}^1 = \sum_{i\in M_1} s_i^{1-1} STPR_i^{1-1} = \left(\frac{1}{2}\times\frac{2}{1}\right)+\left(\frac{1}{2}\times\frac{1}{1}\right) = 1.5$			$LTPR_{EA}^1 = LTPR_{EA}^0 STPR_{EA}^1 = 1\times 1.5 = 1.5$		

In the following period, the two-stage approach to aggregation process will remove all impacts of adding a new item $(i = 3)$ to a price sample – adhering to the 'fixed basket' nature of a price index, while allowing the sampled items to change. The new item has a known expenditure weight and price at time period 1.

Table 3: Elementary Aggregate price sample in the next period – Sample Changes: Removal of item 2 and the addition of item 3, with a known expenditure weight.

i	*Expenditure Weight:* w_i^1	*Expenditure Share:* s_i^1	*Price:* Time = 1	Time = 2	$STPR_i^2$
1	$w_1^1 = 1\times(2/1) = 2$	$s_1^1 = 2/(2+1.5) = 4/7$	$p_1^1 = \$2$	$p_1^2 = \$2$	$STPR_1^2 = 2/2$
2	Removed	Removed	$p_2^1 = \$1$	Removed	Removed
3	$w_3^1 = 1.5$ **(New Item)**	$s_3^1 = 1.5/(2+1.5) = 3/7$	$p_3^1 = \$3$	$p_3^2 = \$2$	$STPR_3^2 = 2/3$

Two-Stage Aggregation:

$$STPR_{EA}^2 = \sum_{i\in M_2} s_i^{2-1} STPR_i^{2-1} = \left(\frac{4}{7}\times\frac{2}{2}\right)+\left(\frac{3}{7}\times\frac{2}{3}\right) = 0.8571$$

$$LTPR_{EA}^2 = LTPR_{EA}^1 STPR_{EA}^2$$

$$LTPR_{EA}^2 = 1.5\times 0.8571 = 1.2857$$

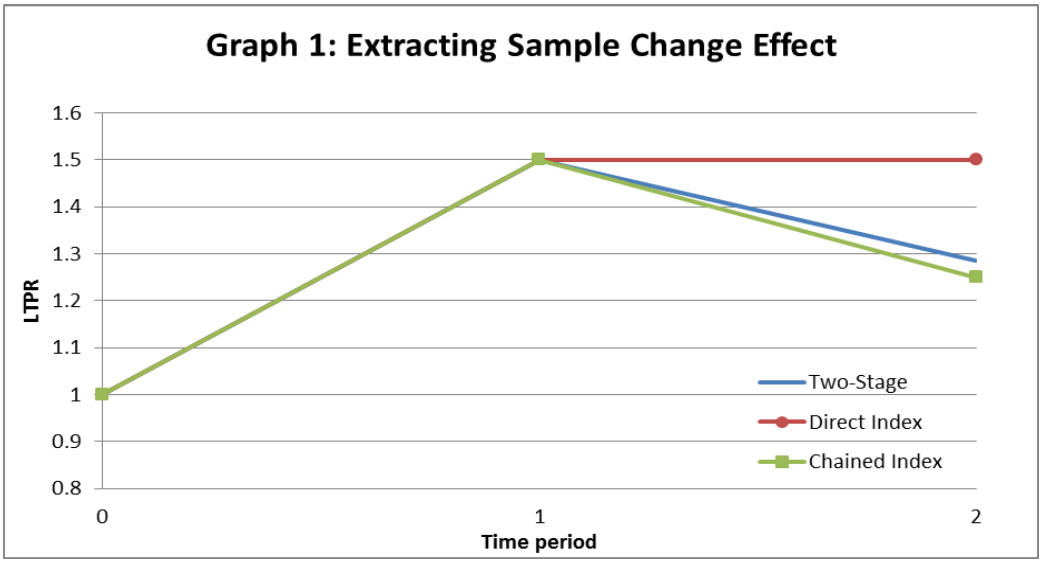
Direct Index: (without index change factor)

$$I_{EA}^2 = \left(0.5\times\frac{2}{1}\right)+\left(0.5\times\frac{2}{2}\right) = 1.5$$

Chained Index:

$$I_{EA}^2 = 1.5\times\left[\left(0.5\times\frac{2}{3}\right)+\left(0.5\times\frac{2}{2}\right)\right] = 1.25$$

With the price updating and rescaling of expenditure weights to form expenditure shares, the effect of sample change does not impact the measure of price change – which occurs in the chained index shown in table 3. This point is further illustrated in graph 1, showing what the direct and chained price index would have been had the sample change not be accounted for.



7. Removing quality change impacts

A key requirement of price indexes is to price to constant quality. The international manuals discuss a range of direct and indirect quality adjustment techniques that can be applied. A key requirement is to ensure that the impact of the quality change does not affect the fixed basket approach to index calculation. Within the index factorisation shown in section 4– equations (2) and (5), the update of expenditure shares was shown to be possible by using either the most recent price relative or the long term price relative. However, if there is a change in quality over time, this will mean the raw price observation change will no longer equal the quality adjusted index movement. In order to capture this change, a quality adjusted short term price relative is used:

Quality Adjusted STPR: $STPR_i^t = \dfrac{p_i^t}{p_i^{t-1} \times Qfactor}$

Where, $Qfactor$ is the proportion of quality change recorded for the item; if no quality change occurs, the $Qfactor$ is equal to 1.

In order to ensure that changes in quality do not cause an index to lose transitivity, the chosen approach when updating expenditure share weights is to use price relatives. In particular, the long term price relative for each observation was chosen, as each price relative is measured on the quality adjusted price movement of each item.

This concept can be seen in the following example of the two approaches to weight update, using a large quality change for illustrative purposes:

Prices update approach: $$s_i^{t-1} = \frac{w_i^{t-1}}{\sum_{i \in M_t} w_i^{t-1}} = \frac{p_i^{t-1} q_i^0}{\sum_{i \in M_t} p_i^{t-1} q_i^0}$$

Price relative update approach: $$s_i^{t-1} = \frac{w_i^{t-1}}{\sum_{i \in M_t} w_i^{t-1}} = \frac{q_i^0 \, p_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} q_i^0 \, p_i^0 LTPR_i^{t-1}}$$

If no quality changes occur, then the price relative updated expenditure share will equal the raw price updated value, as: $p_i^{t-1} = p_i^0 LTPR_i^{t-1}$. When applied to numerical data:

Table 4a: Example of quality change on price sample values

i	***Quantity Weight:*** q_i^0	***Price:*** p_i^t			***Quality Adj. STPR:***	
		Time = 0	**Time = 1**	**Time = 2**	$STPR_i^1$	$STPR_i^2$
1	$q_1^0 = 1$	$p_1^0 = \$1$	$p_1^1 = \$2$* [Qual. Adjust =x2] 100% quality increase = No price change	$p_1^2 = \$2$	$\dfrac{2}{1 \times (2)} = 1$	$\dfrac{2}{2} = 1$
2	$q_2^0 = 1$	$p_2^0 = \$1$	$p_2^1 = \$2$	$p_2^2 = \$1$	$\dfrac{2}{1} = 2$	$\dfrac{1}{2} = 0.5$

The two product price sample returns to the link period prices and the resulting price index must return to its link period value to be transitive. The Calculations below illustrate the outcomes of each price update approach and the benefits of the price relative update approach.

Table 4b: Approaches to index calculation with quality changes

Time period $0 \rightarrow 1$: Index Calculation:

Approach 1: Prices Update	**Approach 2: Price Relative Update**
Expenditure Share:	Expenditure Share:
$s_1^{1-1} = \frac{1\times1}{(1\times1)+(1\times1)} = \frac{1}{2}$ $s_2^{1-1} = \frac{1\times1}{(1\times1)+(1\times1)} = \frac{1}{2}$	$s_i^{1-1} = \frac{[1\times1]\times1}{[1\times(1)]\times1+[1\times(1)]\times1} = \frac{1}{2} \quad i=\{1,2\}$ As, $LTPR_i^0 = 1$
Aggregate STPR & LTPR:	Aggregate STPR & LTPR:
$STPR_{EA}^1 = \left(\frac{1}{2}\right)\times1+\left(\frac{1}{2}\right)\times2 = 1.5$	$STPR_{EA}^1 = \left(\frac{1}{2}\right)\times1+\left(\frac{1}{2}\right)\times2 = 1.5$
(Correct)	***(Correct)***
$LTPR_{EA}^1 = 1\times1.5 = 1.5$	$LTPR_{EA}^1 = 1\times1.5 = 1.5$

Time period $1 \rightarrow 2$: Index Calculation:

Approach 1: Prices Update	**Approach 2: Price Relative Update**
Expenditure Share:	Expenditure Share:
$s_1^{2-1} = \frac{2^*\times1}{(2^*\times1)+(2\times1)} = \frac{1}{2}$ $s_2^{2-1} = \frac{2\times1}{(2^*\times1)+(2\times1)} = \frac{1}{2}$	$s_1^{2-1} = \frac{[1\times(1)]\times1}{[1\times(1)]\times1+[1\times(2)]\times1} = \frac{1}{3}$ $s_2^{2-1} = \frac{[1\times(2)]\times1}{[1\times(1)]\times1+[1\times(2)]\times1} = \frac{2}{3}$
Aggregate STPR & LTPR:	Aggregate STPR & LTPR:
$STPR_{EA}^2 = \left(\frac{1}{2}\right)\times\frac{2}{2}+\left(\frac{1}{2}\right)\times\frac{1}{2} = 0.75$	$STPR_{EA}^2 = \left(\frac{1}{3}\right)\times\frac{2}{2}+\left(\frac{2}{3}\right)\times\frac{1}{2} = \frac{2}{3} = 0.6667$
(Incorrect)	***(Correct)***
$LTPR_{EA}^2 = 1.5\times0.75 = 1.125$	$LTPR_{EA}^2 = 1.5\times0.6667 = 1$

As the example shows, the use of prices to update expenditure weights means that the new item quality is used to calculate the expenditure weights. Alternatively, by using the quality adjusted STPR to update the expenditure weights each period, the fixed basket approach is upheld consistent with calculating the price relative and base period expenditure share. Therefore, the approach recommended here is to use the quality adjusted price relatives method to update expenditure weights and retain the fixed basket expenditure shares.

8. Analytical measures – points contribution

Points contributions is a measure of how much each component contributes to the all-groups price index for the current period, regardless of that component's level in the index.

Points contribution allows for the index to be decomposed into additive components. For example, the points contribution of a COICOP sub-division is made up of the sum of the points contribution from its component COICOP groups, which in turn are made up of the sum of the component classes, which in turn are made up from the points contribution of the elementary aggregates.

Points contribution is a combination of two things: the weight of the component in the link period, and the proportion by which that component's price has changed since the link period.

8.1 Calculation of points contribution and the two-stage approach

Points contributions are calculated by taking the ratio of the current period weight of the component, to the weight of the root index (in general), and multiplying this ratio by the root index number. The root index is the upper level index that the points contribution is measured against.

It can be calculated using either current period values or values from the link period k. The advantage of using the link period k is that points contribution can be calculated before the root level index is aggregated.

$$PC_x^t = I_{P.ROOT}^t \frac{w_x^t}{w_{ROOT}^t}$$

$$PC_x^t = \left(I_{P.ROOT}^k \frac{w_{ROOT}^t}{w_{ROOT}^k} \right) \frac{w_x^t}{w_{ROOT}^t}$$

$$PC_x^t = I_{P.ROOT}^k \frac{w_x^t}{w_{ROOT}^k}$$

An Example of this is seen with the calculation of the points Contribution December quarter 2011 for the Fruit EC in the CPI:

Link Period Approach: $PC_{FRUIT}^{Dec\ 2011} = I_{CPI}^{June\ 2011} \frac{w_{FRUIT}^{Dec\ 2011}}{w_{CPI}^{June\ 2011}} = 99.2 \times \frac{1.37}{100} = 1.36$
Current Period Approach: $PC_{FRUIT}^{Dec\ 2011} = I_{CPI}^{Dec\ 2011} \frac{w_{FRUIT}^{Dec\ 2011}}{w_{CPI}^{Dec\ 2011}} = 99.8 \times \frac{1.37}{100.6} = 1.36$

Converting this procedure to the two-stage approach,

$$PC_x^t = I_{P.ROOT}^k \frac{w_x^t}{w_{ROOT}^k}$$

$$PC_x^t = I_{P.ROOT}^k \frac{w_x^k}{w_{ROOT}^k} \frac{w_x^t}{w_x^k}$$

$$PC_x^t = I_{P.ROOT}^k \frac{w_x^k}{w_{ROOT}^k} \frac{I_{P.x}^{t-1}}{I_{P.x}^k} \frac{I_{P.x}^t}{I_{P.x}^{t-1}}$$

$$PC_x^t = I_{P.ROOT}^k \times s_x^k LTPR_x^{t-1} STPR_x^t$$

For example, CPI Points Contribution December quarter 2011 for the Fruit EC in the CPI:

Two-stage approach – points contribution:

$$PC_{FRUIT}^{Dec\ 2011} = I_{CPI}^{June\ 2011} s_{FRUIT}^{June\ 2011} LTPR_{FRUIT}^{Sept\ 2011} STPR_{FRUIT}^{Dec\ 2011}$$
$$= 99.2 \times 0.016 \times 0.9885 \times 0.8659 = 1.36$$

Where, $s_{FRUIT}^{June\ 2011}$ is the expenditure share of fruit in relation to the root index – which is the CPI all-groups level in this example.

9. Practical Implications for National Statistical Offices

The two-stage approach to aggregation is a robust method of aggregating price values and handling sample changes and quality adjustment. The method does provide some practical challenges, including the development of new business processes and systems. One example of requiring updated business processes relates to the ability to observe updated weights. This may cause analysts to update or change weights based on short run “shocks” – in turn increasing the risk of chain drift and loss of transitivity. In order to account for this, NSOs must maintain proper sample maintenance procedures to mitigate this risk.

The ability to make revisions to price indexes using the two-stage approach is also of interest to the ABS[5]. It’s clear that the two-stage approach to aggregation caters for a two period window, so all revisions must be applied to the period in question and progressively updated in subsequent periods. With the use of clearly defined business

[5] The ABS announced in 2012, as part of the PPI review, that from the September quarter 2014, the PPIs and ITPIs will be revised to accommodate improved data in subsequent quarters. See ABS Cat.no. 6427.0.55.004 - Information Paper: Outcome of the Review of the Producer and International Trade Price Indexes, 2012

processes and system applications, NSOs will be able to incorporate revisions into price indexes.

10. Future work

Further investigation by the ABS into the two-stage approach to aggregation is anticipated in future.

The ABS is particularly keen to undertake an economic assessment of the geometric aggregation at the upper levels of the price indexes. Details of how this might be applied within the two-stage aggregation method are given in Attachment 1.

The table in Attachment 2 shows the derivation of expenditure shares and expenditure weights from a number of price index formulae within the two-stage aggregation context

11. Conclusion

This paper has progressed the intermediate form of the two-stage approach to aggregation to the final practical form. The elementary aggregate long term price relative can simply be updated by the price updated expenditure share and short term price relatives of each price observation. In turn, this enables a two period view of price observations for short term aggregation calculations, which then update a chained longer term elementary aggregate price relative and index – allowing for samples to be changed in a simple manner each period.

Also shown is the use of quality adjusted price relatives to update the expenditure weights and produce price indexes that retain transitivity when item qualities change.

Finally, it is anticipated that the two-stage approach to index aggregation will lead to enabling sample flexibility and more transparent price index aggregation system applications to be developed.

Attachment 1: Geometric application of two-stage approach

The Geometric Laspeyres price index formulae (also the Geometric Lowe and Geometric Young index), can also use the two-stage approach to aggregation. The key difference between the arithmetic and geometric aggregation is that the expenditure weights are not price updated from the link period (t=0) onwards. The Geometric Young index is used as it is consistent with unitary elasticity of substitution *(ILO Consumer Price Index Manual, chapter 1, 1.35, pg. 5)*. As a result, the expenditure weights and expenditure shares for geometric applications will be as follows:

$$w_i^{t-1} = w_i^0; \qquad s_i^{t-1} = \frac{w_i^{t-1}}{\sum_{i \in M_t} w_i^{t-1}} = \frac{w_i^0}{\sum_{i \in M_t} w_i^0};$$ *for all* $t = 1, 2, 3, ...$

The geometric formulae will require the calculation of the short term price relative (STPR) to be calculated geometrically. Following the calculation of the geometric STPR, the index calculation is done as per the arithmetic formulation. The complete formulae are described as follows:

Table 5: Two-stage approach to aggregation: (Geometric formulae)

$$I_{EA(Geo-Laspeyres)}^t = I_{EA}^0 LTPR_{EA}^{t-1} STPR_{EA(Geo)}^t$$

$$STPR_{EA(Geo)}^t = \prod_{i \in M_t} \left(STPR_i^t\right)^{s_i^{t-1}} = \exp\left[\sum_{i \in M_t} s_i^{t-1} \log_e\left(STPR_i^t\right)\right]$$

Where,

$$s_i^{t-1} = \frac{w_i^0}{\sum_{i \in M_t} w_i^0} = \frac{w_i^{t-1}}{\sum_{i \in M_t} w_i^{t-1}}; \quad LTPR_{EA}^{t-1} = LTPR_{EA}^{t-2} STPR_{EA}^{t-1}; \quad LTPR_{EA}^0 = 1; \quad STPR_i^t = \frac{p_i^t}{p_i^{t-1}}$$

For all $i \in M_t$; $t = 1, 2, 3, ...$

As table 6 shows, by taking logarithms of the price relative values, a geometric aggregation can be expressed in an additive form. This form can lead to greater flexibility within systems based on this process.

Attachment 2: Index formulae represented under the two-stage approach to aggregation:

In addition to the sample change abilities of the two-stage approach to aggregation, a core attribute of the method is the ability to harmonise the application of a suite of price indexes used in practice under two distinctive functions – Arithmetic and Geometric. This is done by defining each price index variant used in practice by the expenditure weights allocated to them. Table 6 illustrates the expenditure weights of various price index and elementary aggregate indexes which can be applied within the two-stage aggregation framework.

Table 6: Mapping classical Price index formulae to two-stage approach to aggregation

Price Index Type:	**Expenditure Weights & Expenditure Shares for Two-Stage Aggregation:**
<u>Arithmetic two-stage aggregation:</u> $I^t_{EA(Laspeyres)} = LTPR^{t-1}_{EA} \sum_{i \in M_t} s_i^{t-1} STPR_i^t$	
• **Arithmetic Laspeyres-type Index:** *(in expenditure share and quantity forms):* $I^t_{Laspeyres} = \frac{\sum_{i \in M_t} p_i^t q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} = \sum_{i \in M_t} \frac{p_i^0 q_i^0}{\sum_{i \in M_t} p_i^0 q_i^0} \left(\frac{p_i^t}{p_i^0} \right)$	**Laspeyres:** $w_i^0 = p_i^0 q_i^0$ $s_i^{t-1} = \frac{p_i^0 q_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} p_i^0 q_i^0 LTPR_i^{t-1}}$
Including the Generic Cases of the:	
• **Lowe Index:** $I^t_{Lowe} = \frac{\sum_{i \in M_t} p_i^t q_i^b}{\sum_{i \in M_t} p_i^0 q_i^b} = \sum_{i \in M_t} \frac{p_i^0 q_i^b}{\sum_{i \in M_t} p_i^0 q_i^b} \left(\frac{p_i^t}{p_i^0} \right)$	**Lowe:** $w_i^0 = p_i^0 q_i^b$ $s_i^{t-1} = \frac{p_i^0 q_i^b LTPR_i^{t-1}}{\sum_{i \in M_t} p_i^0 q_i^b LTPR_i^{t-1}}$
• **Young Index:** $I^t_{Young} = \sum_{i \in M_t} \frac{p_i^b q_i^b}{\sum_{i \in M_t} p_i^b q_i^b} \left(\frac{p_i^t}{p_i^0} \right)$	**Young:** $w_i^0 = p_i^b q_i^b$ $s_i^{t-1} = \frac{p_i^b q_i^b LTPR_i^{t-1}}{\sum_{i \in M_t} p_i^b q_i^b LTPR_i^{t-1}}$
• **Carli Index (Average of Price Relatives):** $I^t_{Carli} = \frac{\sum_{i \in M_t} \left(\frac{p_i^t}{p_i^0} \right)}{\lvert M_t \rvert}$	**Carli:** $w_i^0 = 1$ $s_i^{t-1} = \frac{LTPR_i^{t-1}}{\sum_{i \in M_t} LTPR_i^{t-1}}$
• **Dutot Index (Ratio of Average Price):** $I^t_{Dutot} = \frac{\sum_{i \in M_t} p_i^t \Big/ \lvert M_t \rvert}{\sum_{i \in M_t} p_i^0 \Big/ \lvert M_t \rvert} = \frac{\sum_{i \in M_t} p_i^0 \left(\frac{p_i^t}{p} \right)}{\sum_{i \in M_t} p_i^0}$	**Dutot:** $w_i^0 = p_i^0$ $s_i^{t-1} = \frac{p_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} p_i^0 LTPR_i^{t-1}}$

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<u>Geometric two-stage aggregation:</u>

$$I^t_{EA(Geo-Laspeyres)} = I^0_{EA} LTPR^{t-1}_{EA} \prod_{i \in M_t} \left(STPR^t_i \right)^{s^{t-1}_i}$$

Where,

$$STPR^t_{EA(Geo)} = \prod_{i \in M_t} \left(STPR^t_i \right)^{s^{t-1}_i} = \exp\left[\sum_{i \in M_t} s^{t-1}_i \log_e \left(STPR^t_i \right) \right]$$

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- **Geometric Laspeyres (also Young and Lowe) Index:**

$$I^t_{Laspeyres} = \prod_{i \in M_t} \left(\frac{p^t_i}{p^0_i} \right)^{s^{t-1}_i} \qquad s^{t-1}_i = \frac{p^0_i q^0_i}{\sum_{i \in M_t} p^0_i q^0_i}$$

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Geometric Laspeyres:

$$w^{t-1}_i = p^0_i q^0_i$$

$$s^{t-1}_i = \frac{p^0_i q^0_i}{\sum_{i \in M_t} p^0_i q^0_i}$$

For all $t = 0,1,2,....$

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- **Jevons Index:**

(Equally Weighted Geometric average of price relatives)

$$I^t_{Jevons} = \prod_{i \in M_t} \left(\frac{p^t_i}{p^0_i} \right)^{1/|M_t|}$$

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Jevons:

$$w^t_i = 1$$

$$s^t_i = \frac{1}{|M_t|}$$

For all $t = 0,1,2,....$

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Appendix

1. Notation Used Within Report:

$I_X^t =$	Aggregate Price index at time t, where X can be an Elementary Aggregate (EA) or Upper level index.
$p_i^t =$	Price of product i at time t
$q_i^t =$	Quantity of product i at time t
$M_t =$	Set of products within the price sample at time t
$STPR_i^t =$	Short term price relative of product i (or EA) at time t
$LTPR_i^t =$	Long term price relative of product i (or EA) at time t
$w_i^0 =$	Link period expenditure weight of product i (or EA) at time t
$w_i^t =$	Price updated expenditure weight of product i (or EA) at time t
$s_i^0 =$	Link period expenditure share of product i (or EA) at time t
$s_i^t =$	Price updated expenditure share of product i (or EA) at time t (also known as a hybrid expenditure share)
$PC_x^t =$	Points Contribution of EA x (or Root level) at time t

2. Aggregate Relationship Derivation:

In order to derive the aggregate expenditure relationship shown in equation (4) within the paper: $s_i^{t-1} LTPR_{EA}^{t-1} = s_i^0 LTPR_i^{t-1}$, the following properties must apply:

Let:

- The link period expenditure share: $s_i^0 = \dfrac{w_i^0}{\sum_{i \in M_t} w_i^0}$;
- The Elementary Aggregate long term price relative: $LTPR_{EA}^{t-1} = \sum_{i \in M_t} s_i^0 LTPR_i^{t-1}$; where $LTPR_i^{t-1} = STPR_i^1 STPR_i^2 \ldots STPR_i^{t-1}$;
- Price updated and rescaled expenditure shares (also known as hybrid expenditure shares): $s_i^{t-1} = \dfrac{w_i^0 LTPR_i^{t-1}}{\sum_{i \in M_t} w_i^0 LTPR_i^{t-1}}$; and
- The Constraint at any period that the sum of all a price samples expenditure shares must equal 1, ie. $\sum_{i \in M_t} s_i^{t-1} = 1$

Therefore the following must hold:

$$s_i^{t-1}LTPR_{EA}^{t-1} = \frac{w_i^0 LTPR_i^{t-1}}{\sum_{i\in M_t} w_i^0 LTPR_i^{t-1}} \sum_{i\in M_t} \frac{w_i^0}{\sum_{i\in M_t} w_i^0} LTPR_i^{t-1}$$

$$s_i^{t-1}LTPR_{EA}^{t-1} = \frac{w_i^0 LTPR_i^{t-1}}{\sum_{i\in M_t} w_i^0 LTPR_i^{t-1}} \sum_{i\in M_t} \frac{w_i^0 LTPR_i^{t-1}}{\sum_{i\in M_t} w_i^0}$$

$$s_i^{t-1}LTPR_{EA}^{t-1} = \frac{w_i^0 LTPR_i^{t-1}}{\sum_{i\in M_t} w_i^0}$$

$$s_i^{t-1}LTPR_{EA}^{t-1} = \frac{w_i^0}{\sum_{i\in M_t} w_i^0} LTPR_i^{t-1}$$

$$s_i^{t-1}LTPR_{EA}^{t-1} = s_i^0 LTPR_i^{t-1} \qquad \forall i \in M_t$$

Intuitively, this outcome describes the mechanism of price updating and rescaling (normalisation), with each i^{th} product price movement divergence from the aggregate measure being captured by the price updated expenditure share.

3. Sample change – Proving Sample change maintains a Laspeyres Index series:

As shown within the paper, the two stage approach to aggregation enables the Laspeyres index produced to retain price index transitivity when price samples change. In order to formally show the price index back series is unchanged with the addition of new items within the price sample under the conditions presented in section 6 of the report, the following can be seen:

If we have an elementary aggregate price index:

$$I_{EA}^t = LTPR_{EA}^{t-1}STPR_{EA}^t = STPR_{EA}^0 STPR_{EA}^1 ... STPR_{EA}^t$$

If at time $t = b$, a new set of items E_b has been added to the price sample M_{b-1}

such that $M_{b-1} = M_b \setminus E_b$

Then, in order to show that the back series index,

$$I_{EA}^{b-1} = STPR_{EA}^0 STPR_{EA}^1 ... STPR_{EA}^{b-1}$$

has not changed, we must show that for the arithmetic Laspeyres price index:

$$\sum_{i\in M_b} s_i^{b-x-1} STPR_i^{b-x} = STPR_{EA}^{b-x} \qquad \forall x : x \in \{1,..,b-1\} \qquad \text{(a)}$$

Using sample change rules (see section 6), we know that for all $j \in E_b$:

$$STPR_j^{b-x} = STPR_{EA}^{b-x} \quad ; \quad LTPR_j^{b-x} = LTPR_{EA}^{b-x} \quad ; \quad w_j^0 = \frac{w_j^{b-x}}{LTPR_{EA}^{b-x}}$$

If we expand the sum of equation (a):

$$\sum_{i\in M_b} s_i^{b-x-1} STPR_i^{b-x}$$

$$= \sum_{i\in M_{b-1}} s_i^{b-x-1} STPR_i^{b-x} + \sum_{j\in E_b} s_j^{b-x-1} STPR_i^{b-x}$$

$$= \sum_{i\in M_{b-1}} s_i^{b-x-1} STPR_i^{b-x} + STPR_{EA}^{b-x} \sum_{j\in E_b} s_j^{b-x-1}$$

$$= \sum_{i\in M_{b-1}} \frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} STPR_i^{b-x} + STPR_{EA}^{b-x} \sum_{j\in E_b} \frac{w_j^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}}$$

$$= \frac{1}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} \left[\sum_{i\in M_{b-1}} w_i^{b-x-1} STPR_i^{b-x} + STPR_{EA}^{b-x} \sum_{j\in E_b} w_j^{b-x-1} \right] \qquad \text{(b)}$$

If we then multiply equation (b) by: $\dfrac{\sum_{i\in M_{b-1}} w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}$; we'll get:

$$\sum_{i\in M_b} s_i^{b-x-1} STPR_i^{b-x}$$

$$= \frac{1}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} \left[\sum_{i\in M_{b-1}} w_i^{b-x-1} STPR_i^{b-x} + STPR_{EA}^{b-x} \sum_{j\in E_b} w_j^{b-x-1} \right] \frac{\sum_{i\in M_{b-1}} w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}$$

$$= \frac{\sum_{i\in M_{b-1}} w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} \left[\sum_{i\in M_{b-1}} \frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}} STPR_i^{b-x} + STPR_{EA}^{b-x} \frac{\sum_{j\in E_b} w_j^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}} \right]$$

$$= \frac{\sum_{i\in M_{b-1}} w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} \left[STPR_{EA}^{b-x} + STPR_{EA}^{b-x} \frac{\sum_{j\in E_b} w_j^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}} \right]$$

$$= STPR_{EA}^{b-x} \left(\frac{\sum_{i\in M_{b-1}} w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}} \left[\frac{\sum_{i\in M_{b-1}} w_i^{b-x-1} + \sum_{j\in E_b} w_j^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}} \right] \right)$$

$$= STPR_{EA}^{b-x}$$

$$\therefore \sum_{i\in M_b} s_i^{b-x-1} STPR_i^{b-x} = STPR_{EA}^{b-x} \qquad \forall x : x \in \{0,1,\ldots,b-1\}$$

Therefore the addition of price observations to the price sample using the rules described in section 6 results in no changes to the price index back series, thus upholding the Laspeyres price index transitivity property for the two stage approach to price index aggregation.

4. Sample change – Proving Sample change maintains a Geometric Laspeyres Index series:

Similar to appendix 3, the following will show that the two stage approach to aggregation also maintains geometric Laspeyres price index transitivity. This can be shown as follows:

If we have an elementary aggregate price index:

$$I_{EA}^{t} = LTPR_{EA}^{t-1} STPR_{EA}^{t} = STPR_{EA}^{0} STPR_{EA}^{1} ... STPR_{EA}^{t}$$

If at time $t = b$, a new set of items E_b has been added to the price sample M_{b-1} such that $M_{b-1} = M_b \setminus E_b$

Then, in order to show that the back series index,

$$I_{EA}^{b-1} = STPR_{EA}^{0} STPR_{EA}^{1} ... STPR_{EA}^{b-1}$$

has not changed, we must show that for the geometric Laspeyres price index:

$$\prod_{i \in M_b} \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right)^{s_i^{b-x-1}} = STPR_{EA}^{b-x} \qquad \forall x : x \in \{1,..,b-1\} \qquad \text{(c)}$$

Using sample change rules (see section 6), we know that for all $j \in E_b$, in the geometric case:

$$STPR_j^{b-x} = STPR_{EA}^{b-x} \quad ; \qquad LTPR_j^{b-x} = LTPR_{EA}^{b-x}$$

If we expand the product of equation (c):

$$\prod_{i \in M_b} \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right)^{s_i^{b-x-1}}$$

$$= \exp\left[\sum_{i \in M_b \setminus E_b} s_i^{b-x-1} \log_e \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right) + \sum_{i \in E_b} s_i^{b-x-1} \log_e \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right) \right]$$

$$= \exp\left[\sum_{i \in M_b \setminus E_b} s_i^{b-x-1} \log_e \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right) + \sum_{i \in E_b} s_i^{b-x-1} \log_e \left(STPR_{EA}^{b-x} \right) \right]$$

$$= \exp\left[\sum_{i \in M_b \setminus E_b} s_i^{b-x-1} \log_e \left(\frac{p_i^{b-x}}{p_i^{b-x-1}} \right) \right] \left(STPR_{EA}^{b-x} \right)^{\sum_{j \in E_b} s_j^{b-x-1}} \qquad \text{(d)}$$

If we apply the following facts to equation (d):

$$M_{b-1} = M_b \setminus E_b \qquad ; \qquad \frac{s_i^{b-x-1}}{s_i^{b-x-1}} = \frac{\dfrac{w_i^{b-x-1}}{\sum_{i \in M_{b-1}} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i \in M_{b-1}} w_i^{b-x-1}}} = 1$$

We get the following:

$$\prod_{i\in M_b}\left(\frac{p_i^{b-x}}{p_i^{b-x-1}}\right)^{s_i^{b-x-1}}$$

$$=\exp\left[\sum_{i\in M_{b-1}}\frac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}\log_e\left(\frac{p_i^{b-x}}{p_i^{b-x-1}}\right)\right]\left(STPR_{EA}^{b-x}\right)^{\sum_{j\in E_b} s_j^{b-x-1}}$$

$$=\exp\left[\sum_{i\in M_{b-1}}\frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}\log_e\left(\frac{p_i^{b-x}}{p_i^{b-x-1}}\right)\right]\left(STPR_{EA}^{b-x}\right)^{\sum_{j\in E_b} s_j^{b-x-1}}$$

$$=\exp\left[\sum_{i\in M_{b-1}}\frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}\log_e\left(\frac{p_i^{b-x}}{p_i^{b-x-1}}\right)\right]\exp\left[\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}\right]\left(STPR_{EA}^{b-x}\right)^{\sum_{j\in E_b} s_j^{b-x-1}}$$

$$=\exp\left[\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}\log_e\left(STPR_{EA}^{b-x}\right)\right]\left(STPR_{EA}^{b-x}\right)^{\sum_{j\in E_b} s_j^{b-x-1}}$$

$$=\exp\left[\left(\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}+\sum_{j\in E_b}\frac{w_j^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}\right)\log_e\left(STPR_{EA}^{b-x}\right)\right] \qquad \text{(e)}$$

Therefore, in show (c) is true, we must show:

$$1=\left(\frac{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}}{\dfrac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}}+\sum_{j\in E_b}\frac{w_j^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}\right) \qquad \text{(f)}$$

Thus, expanding the terms of (f) we find that we find the following:

$$\frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}=\frac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}+\frac{w_i^{b-x-1}}{\sum_{i\in M_{b-1}} w_i^{b-x-1}}\sum_{j\in E_b}\frac{w_i^{b-x-1}}{\sum_{i\in M_b} w_i^{b-x-1}}$$

$$\frac{w_i^{b-x-1}}{\sum\limits_{i\in M_b} w_i^{b-x-1}} = \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}} - \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}} \sum_{j\in E_b} \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_b} w_i^{b-x-1}}$$

$$w_i^{b-x-1}\left(\frac{1}{\sum\limits_{i\in M_b} w_i^{b-x-1}}\right) = \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}}\left[1-\sum_{j\in E_b}\frac{w_i^{b-x-1}}{\sum\limits_{i\in M_b} w_i^{b-x-1}}\right]$$

$$w_i^{b-x-1}\left(\frac{1}{\sum\limits_{i\in M_b} w_i^{b-x-1}}\right) = \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}}\left(\frac{1}{\sum\limits_{i\in M_b} w_i^{b-x-1}}\right)\left[\sum_{i\in M_b} w_i^{b-x-1} - \sum_{j\in E_b} w_i^{b-x-1}\right]$$

$$w_i^{b-x-1} = \frac{w_i^{b-x-1}}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}}\left[\sum_{i\in M_b} w_i^{b-x-1} - \sum_{j\in E_b} w_i^{b-x-1}\right] \qquad \text{(g)}$$

If we know that a key fact for all items $i \in M_b$ of Laspeyres indexes is that the link period (t=0) expenditure weight remains fixed, $w_i^0 = w_i^{t-x} \quad \forall x : x \in \{1,2,...,t\}$

Substituting this relationship into (g),we get:

$$1 = \frac{1}{\sum\limits_{i\in M_{b-1}} w_i^{b-x-1}}\left[\sum_{i\in M_b} w_i^{b-x-1} - \sum_{j\in E_b} w_i^{b-x-1}\right]$$

$$\sum_{i\in M_{b-1}} w_i^{b-x-1} = \left[\sum_{i\in M_b} w_i^{b-x-1} - \sum_{j\in E_b} w_i^{b-x-1}\right]$$

$$\sum_{i\in M_{b-1}} w_i^{b-x-1} = \left[\sum_{i\in M_{b-1}} w_i^{b-x-1}\right]$$

Thus, as we've shown (f) is true for all x, equation (e) becomes:

$$\prod_{i\in M_b}\left(\frac{p_i^{b-x}}{p_i^{b-x-1}}\right)^{s_i^{b-x-1}} = \exp\left[(1)\log_e\left(STPR_{EA}^{b-x}\right)\right] = STPR_{EA}^{b-x}$$

Thus the addition of new products to a geometric Laspeyres price index according to the rules specified in section 6 of the report will not impact the price index back series, for all values of $x \in \{1,2,...,b-1\}$.