A recursive Jevons formulation

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Abstract

The unweighted geometric mean value, denoted Jevons index for elementary aggregates, may be applied either to a fixed base or to a rolling base price index with backwards chaining. It is here suggested a Jevons formulated as a recursive combination of the two alternatives in order to accommodate a changing product universe as well as honoring a fixed base, simultaneously within the year. The principle of direct comparisons is adhered to through an exponential parameter combining the simultaneous baskets.

Keywords: Jevons, recursion, fixed base, rolling base

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1 Introduction

Over the past two decades, there has been an accentuated desire of using census-like data in CPI baskets. Come scanner data, standard index methodology has been either adapted or obsolete, opening for elaboration through innovative methods. Although obvious benefits from censuses, which in effect is a new feature, no clarity exists on how to preferentially deal simultaneously with parameters like price, product churn and current quantity weights. This ideal new data reality appears not too ideal to handle in practice. In this note, a suggestion is made for applying the most standard elementary aggregation formula, the Jevons, recursively to exploit the full product universe. Jevons is here formulated as partly fixed and partly chained through a recursively incorporated fixed base.

2 A recursive Jevons formulation

The principle of the fixed base formulation is a comparison of the *latest* occurrence with the *first* occurrence of included products – given a fixed month like December the preceding year for observing *first* occurrences in. In contrast, a monthly chaining formulation is a comparison of *latest* occurrence with the *last* occurrence, running throughout the year and fixated to December the preceding year. In a non-changing product universe with constant weights, the two formulations coincide as the chaining collapses into the first and last occurrences, i.e. a direct comparison. Emphasis can be on either of the two methods regarding which intersecting universe of products to use in standard applications of Jevons. The suggestion herein adheres principally to a fixed basket regime with the amendment of multiple fixed baskets with merely one chained link to a fixed base index each, due to provenance.

2.1 The product universe evolvement over thirteen months

Denote a product group *c* consisting of items *g:* $g \in c$. This is the product universe U_c. By adding a time dimension, at base period *0*, say December year *y-1*, the current items are g^0.

The following month, January year *y*, the universe is g^1. The intersection universe is $g^{1,0}$ and the first amended universe is $g^{1,\overline{0}}$, i.e. items added to the universe with elapse of one time point. The new universe is $U_c^1 = g^{1,0}+g^{1,\overline{0}}$. Items that existed in g^0 and no longer exist in g^1 are in the null subset $g^{\emptyset}$ and these need no further treatment.

In the third month, February, the intersection universe is $g^{2,0}$, the second amended universe is $g^{2,\overline{0}}$ from the *current* month and $g^{2|1,\overline{0}}$ from the previous month for items still available in the current month and that originated in January. Items in the null subset are again not considered. The new universe is $U_c^2 = g^{2,0}+g^{2,\overline{0}}+ g^{2|1,\overline{0}}$.
To emphasize, the amendment $g^{2,\overline{0}}$ (or any $g^{t,\overline{0}}$) comprises items that first occurred in the current month *t* and $g^{2|1,\overline{0}}$ comprises current items that occurred in the conditionally given month (January in this case) but were not in the base, hence $\overline{0}$. To ease notation, $g^{2|1,\overline{0}}$ can be expressed as $g^{2|1}$, implicitly stating that genesis is in period 1, and $g^{2|\overline{0}}$ implicitly states genesis in the current period, period 2, meaning that $g^{2|\overline{1},\overline{0}}$ is equivalent to $g^{2|\overline{0}}$.

Also, note that items in $g^{2,0}$, the intersection of February and the base, may or may not have been in $g^{1,0}$ – implicit conditioning is on existence in the base g^{0}. The pattern is formalized in Table 1.

Table 1 Subsets of a changing product universe

Period	Intersection	Amendment	Total Universe
0	g^{0}	-	$U_c^0 = g^0$
1	$g^{1,0}$	$g^{1,\bar{0}}$	$U_c^1 = g^{1,0} + g^{1,\bar{0}}$
2	$g^{2,0}$	$g^{2,\bar{0}}$, $g^{2\|1,\bar{0}}$	$U_c^2 = g^{2,0} + g^{2,\bar{0}} + g^{2\|1,\bar{0}}$
3	$g^{3,0}$	$g^{3,\bar{0}}$, $g^{3\|2,\bar{0}}$, $g^{3\|1,\bar{0}}$	$U_c^3 = g^{3,0} + g^{3,\bar{0}} + g^{3\|2,\bar{0}} + g^{3\|1,\bar{0}}$
4	$g^{4,0}$	$g^{4,\bar{0}}$, $g^{4\|3,\bar{0}}$, $g^{4\|2,\bar{0}}$, $g^{4\|1,\bar{0}}$	$U_c^4 = g^{4,0} + g^{4,\bar{0}} + g^{4\|3,\bar{0}} + g^{4\|2,\bar{0}} + g^{4\|1,\bar{0}}$
⋮	⋮	⋮	⋮
t>1	$g^{t,0}$	$g^{t,\bar{0}}$, $\sum_{k=1}^{K=t-1} g^{t\|t-k,\bar{0}}$	$U_c^t = g^{t,\bar{0}} + \sum_{k=1}^{K=t-1} g^{t\|t-k,\bar{0}}$

The evolvement of the product universe over time is seen outlined in Table 1, with a growing number of possible amendment sets $g^{t|k,0}$ for time points *k* between the base and current period *t*. Note that this does not necessarily imply a growing set of *actual* items when summarizing the amendment sets. It can be realized that amendment sets are all conditional on respective *first* occurrence after the initial basket base period. For instance, $g^{3|1,\bar{0}}$ reflects items in March that first occurred in January, whereas $g^{3|2,\bar{0}}$ reflects items in March that first occurred in February. This is not equivalent to $g^{3|2,1,\bar{0}}$, which is the intersection universe of items that existed both in January and February as well as March, while not in the base.[1]

2.2 A recursive Jevons formulation

The standard Jevons for the fixed basket formulates, after first comparison period (*t=1*), with a recursive index amendment R_t as

$$IX_t = \left[\frac{P_t}{P_0}\right]^q \times [R_t]^{(1-q)},\ t>1 \qquad (1)$$

following standard notation, prices in the current period *t* and the base period, p_i^t and p_i^0 respectively, are multiplied $\prod_{i=1}^{i=n(g^{t,0})} \left(\frac{p_i^t}{p_i^0}\right)$ to obtain $\frac{P_t}{P_0}$. No item subscripting is needed henceforth and the number

[1] This is path-independency: what happens with items between comparison months does not affect the direct comparison principle adhered to here.

of items from the intersecting universe is denoted $n(g^{t,0})$. It will be seen that (1) becomes a recursive statement for *t*>1.

The relative importance between the fixed basket Jevons part and the recursive Jevons amendment R_t is regulated through *q*, *q*≤*1*. The choice of *q* requires elaboration as it replaces the application of expenditure shares, hence the first part of the right hand side in (1) is a simultaneous proxy for a geometric Laspeyres (§16.75,§22.32 in ILO 2004) and a geometric Paasche (ibid. §16.76, §16.80) given the fixed base. A dynamic *q* based on past as well as most recent observations of expenditures, in say *t*, may result in an uncontrolled drifting index due to the decay rate in the recursive formulation. The choice should be perhaps policy based and/or deterministic throughout the index year.

In period t=1, R_t is by definition null, as outlined in the previous subsection. As of period t=2, the recursive amendment formulates as

$$R_t = \left(\prod_{k=0}^{K=t-2}\left(\frac{P_{t|1+k}}{P_{1+k|1+k}} \times IX_{1+k}\right)\right)^{1/(K+1)}, \; t>1. \qquad (2)$$

It is seen from (2) that each recursive component included in the root expression on the right hand side is weighted equally with *1/K+1*. All new baskets occurring after initial base period 0 are valued equally – however each recursive component comprises all previous recursive components embedded in IX_{1+k}, as given in (1), hence (2) renders a decaying impact from basket *changes* with time.

The recursive amendment R_t (2) applies to the pattern outlined in Table 1 accordingly through the following example for the first two periods after January, *t>1*; *t=2* and *t=3*:

in *t=2*; $R_2 = \sqrt[1]{\frac{P_{2|1,\bar{0}}}{P_{1|1,\bar{0}}} \times \left(\frac{P_1}{P_0} = IX_1\right)}$ for $g^{2|1,\bar{0}}$, and

in *t=3*; $R_3 = \sqrt[2]{\left(\left(\frac{P_{3|2,\bar{0}}}{P_{2|2,\bar{0}}} \times IX_2\right) \times \left(\frac{P_{3|1,\bar{0}}}{P_{1|1,\bar{0}}} \times IX_1\right)\right)}$ for $\left(g^{3|2,\bar{0}} \cup g^{3|1,\bar{0}}\right)$.

It is again seen that R_t will not benefit immediately from the entire amendment – the current period amendment $g^{t,\bar{0}}$ is not used until the immediately following time period when it has its first observable price development, $g^{t|1+k}$ and then consecutively from *k*=(0,1..,*K*=t-2). It is then multiplied with its corresponding recursive formulation IX_{1+k} relating it to the base.

2.3 Operational remarks

In one sense, the suggested recursive Jevons formulation is a group mean imputation method in the amendment part, depending on the choice of q, to fill up the gap between the fixed base and up until first occurrence. And, the principal difference from a forward-carried base price imputation is the temporary disregard of the initial occurrence $g^{t|\bar{0}}$: upcoming items are assigned their own price development in the second month after first occurrence, analogous to a monthly chaining and resampling (MCR) strategy in which items hibernate during the month in which they appear for the first time ever.

However, the approach is not a monthly chaining practice as there is no path dependency; all baskets are fixed to their own introduction month and not to intermittent months. Following notation found in de Haan (2001) for the set of items available in the base month *0* and comparison month t, $I^{0t} = I^0 \cap I^t$, but with mutually exclusive sets with respective base periods $I^{bt} = I^b \cap I^t$ for $b>0$. The method is however not memoryless: at all steps, there is a memory of intermediary time period's indices, similar to a monthly chaining approach, but intermediate months are downplayed through the regulatory q parameter.

3 Setting *q*: quality adjustment regime

It can readily be seen that the recursive Jevons is a combination of one definite, non-changing (but possibly shrinking) basket and all eligible *new* baskets occurring after the base period. A notable but yet outstanding topic when choosing q is the valuation/assessment of the changing item universe that enters the basket/index formulation through the recursive multiplicative amendment. The changing universe may embed implicit price changes due to quality or quantity changes of re-appearing items, i.e. relaunches, besides completely new items i.e. subject to hidden price changes as discussed by de Haan (2001). As may be apparent from (2) no quantity or quality adjustments are accounted for in the setting, which is intentional: each new aggregate is valued with q.

A specific problem is the correct accounting for *relaunches,* i.e. items that re-occur under some different shape and size, hence changing from the original fixed basket to the set of new items within the year. Some approaches, like the QU method by Chessa (2015) deal with this implicitly, or does e.g. the GUV method by Auer (2011) explicitly account for units measured, hence fully including every change in per-net-mass unit price change, also suggested by de Haan (2001) as a *quality-adjusted unit value*. On the other hand, the "extremum" estimator MCR[2] bypasses the issue completely whereas the manually limited fixed basket approach requires manual adjustments for item changes. All known approaches towards the QA issue can be questioned – even the propagated approach of unit-by-unit proportional quantity adjustment in the CPI Manual (ILO, 2004) which is a simplified approach on modeling consumer utility. A similar reservation applies to hedonic adjustments for quality or quantity (ibid. §7.79 et seq.), which usually rely on rather simple regression modeling than more informed modeling.

4 Numerical illustration

In the following example, the artificial dataset in Table 19.1 from the CPI Manual (ILO, 2004) is used. These prices are considered as reflecting items from one homogeneous product group, given in Table 2 with starting with period 1.

Table 2 Artificial prices for six products

Period t	p1	p2	p3	p4	p5	p6
1	1.0	1.0	1.0	1.0	1.0	1.0
2	1.2	3.0	1.3	0.7	1.4	0.8
3	1.0	1.0	1.5	0.5	1.7	0.6
4	0.8	0.5	1.6	0.3	1.9	0.4
5	1.0	1.0	1.6	0.1	2.0	0.2

Source: CPI Manual (ILO, 2004), Table 19.1.

Regarding item life cycles, Table 2 reflects items observable at all time periods (t=1 to 5), i.e. a non-changing universe. The observability associated with this complete item universe can be illustrated as a matrix with unit values reflecting existence at a specific time point. Shown in Table 3, two cases are illustrated; the case in Table 2 and an example of a changing universe with zeros indicating non-existence.

[2] This appears to be, in practice, the greediest possible way of including the full product universe at any time and circumvents all manual intervention requirements.

Table 3 Observability matrices L1 and L2

Period t	L1	p1	p2	p3	p4	p5	p6	L2	p1	p2	p3	p4	p5	p6
1		1	1	1	1	1	1		1	1	1	0	0	0
2		1	1	1	1	1	1		1	1	1	1	1	0
3		1	1	1	1	1	1		1	1	1	1	1	1
4		1	1	1	1	1	1		1	1	1	1	1	1
5		1	1	1	1	1	1		1	1	0	0	1	0

The second part of Table 3, L2, reflects an observability situation of items over time in which merely products 1 and 2 exist throughout the entire time span. Products 4 and 5 first occur in time point two and product 6 first occurs in time point three. Correspondingly, products 3, 4 and 6 are unobservable in time point six. Applying the two observability regimes L1 and L2 on the data in Table 2, the outcome from the recursive Jevons is given in Table 4, together with comparison indexes.

Table 4 Index outcomes from observabilities in Table 3 (L1 & L2) for the fixed base (FB), monthly chaining (MCR), the recursive amendment R and the fixed base with recursive amendment, FBR.

Period t	FB,L1	MCR,L1	FB, L2	MCR, L2	R, L2	FBR, L2
2	1.242	1.242	1.673	1.673		1.673
3	0.956	0.956	1.145	1.215	1.558	1.162
4	0.726	0.726	0.862	0.922	0.994	0.868
5	0.632	0.632	1.000	1.272	1.240	1.011

Note: Period 1 is base with unit prices (=1), as seen in Table 2. Indices are divided by 100.

Judging from the simplified and limited outcomes in Table 4, the recursive approach (FBR, L2) can be seen to dampen the effects from the updated basket, observed for the monthly chaining (MCR), as well as the contribution (R, L2) is added (multiplicatively) to the fixed base Jevons (FB, L2).

5 Remarks on the recursive Jevons

The suggested method is formulated so far without any lowest-level weights, i.e. included prices are unweighted. This remains to be elaborated on. However, in the present setting and if relying on a size-proportional sampling, the approach should operate similar to a standard Jevons implementation.

The recursive Jevons is here assumed to fulfill the same axiomatic index tests that are met by the standard Jevons formulation. However, there is a caveat from the q exponent in the formulation – each component in (1) and (2) are Jevons formulations whereas one enters as a recursive multiplicative component. As noted by de Haan (2001), axiomatic tests were based on a fixed set of goods with constant observability over time, hence it remains to analyze the suggested formulation in axiomatic terms.

6 Concluding notes

The approach presented herein is not elaborated on deeply but rather a conceptual suggestion – it is a simple and transparent way of re-using the fixed base Jevons to account for a changing product universe, instead of applying more complicated index methodology.

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