Seasonal Products and Multilateral Methods

Botir Radjabov and Ken Van Loon[1]

Abstract:

Which method to prefer for seasonal products remains one of the most difficult choices in price statistics. This research uses seasonal scanner data samples to compare and to discuss traditional bilateral methods and the most common multilateral methods to find out which of these methods could be preferred for seasonal products. This research shows that all traditional bilateral methods have disadvantages and should not be used for seasonal products. It also shows that GEKS method rather than GK or WTPD methods can be considered as a somewhat preferred method for seasonal products if proper GEKS calculations are possible. It shows as well that if proper GEKS calculations are impossible due to no bilateral product matches, ITGEKS method might be seen an alternative to GEKS method. To test the problem of no bilateral product matches for GEKS method, this research proposes several new tests.

[1] The authors would like to thank Ludwig von Auer, Claude Lamboray and Lucien May for their useful comments and suggestions

1. Introduction

Increasing usage of scanner data makes it possible to opt for superlative and multilateral methods for calculations of price indices. However, an increase in a variety of methods does not simplify the decision which method should be preferred in practice. Arguably, this decision is easier to make for non-seasonal rather than for seasonal products. This is since several plausible methods provide similar price indices for the former but not for the latter products. Therefore, this research compares and discusses traditional bilateral methods and the most common multilateral methods to find out which of these methods could be preferred for seasonal products.

The structure of this research is as follows. Section 2 provides basic information regarding seasonal products and discusses the incentives to use multilateral methods for them. Section 3 describes traditional bilateral methods used for seasonal products. Section 4 outlines the most common multilateral methods and discusses some of their properties. Section 5 provides a description of seasonal scanner data samples. Section 6 presents empirical results and proposes several new tests. Section 7 provides a conclusion.

2. Seasonal Products

All seasonal products can be roughly divided into strongly and weakly seasonal products. The former products are only available in a market during a specific season. The latter products are available in a market during the whole year. However, these products experience regular fluctuations in prices and quantities in accordance with specific seasons (ILO et al. 2020).

It is widely accepted that strongly seasonal products create difficulties for compiling price indices. This is due to the following fact. If prices are only available during 1 month, calculations of the relative prices are impossible. Nevertheless, there are several techniques, which might still make such calculations possible (ILO et al. 2020, chapter 11). These techniques are based on price imputations for missing products. The most common price imputation techniques are an overall mean and a price carry forward techniques. It seems that a usage of the former technique is more justified. This is since the price carry forward technique assumes price stability for products when their last available prices are being carried forward. This assumption is usually violated in practice, especially for seasonal products. It is undoubtedly possible not to use any of price imputation techniques at all. However, if there are no price imputations, the price difference between the last set and the next set of available prices of previously missing products is missed.

Describing an overall mean technique, the following facts are noteworthy. This technique uses an average of matched products` price relatives of 2 months to make price imputations (ILO et al. 2020, Chapter 11). This implies that the technique "does not harm" a matched products` part of a price index. Simultaneously, it also makes a price comparison for previously missing products possible. A price comparison which is based on imputed prices makes the technique self-correcting when real prices can be observed again. Given the above advantages, this technique is used for fixed weights price indices in this research.

Before proceeding to the next section, it is beneficial to establish a connection between seasonal products and multilateral methods. To start with, traditional bilateral methods used for seasonal products seem not to work particularly well. The reasons for this are frequent absence of seasonal products in a market and their changing seasons of availability over years. These reasons are further discussed in Section 3. It is arguable that frequent chaining of price indices might be seen as a solution. However, frequent chaining of superlative price indices might lead to “chain drift”, which might occurs due to high entry and low exit prices of seasonal products (ILO et al. 2004). To avoid “chain drift” while accounting for a high number of missing products, multilateral methods are an obvious choice to consider.

3. Bilateral Fixed Base Year Methods

Bean and Stine Type C or Rothwell Method

Bean and Stine Type C (1924) or Rothwell (1958) method (hereinafter Rothwell) uses a vector of seasonal quantities in a month *m*, where *m = 1,2,…,12,* of a base year *0*, $q^{0,m}$. Moreover, it also uses a vector of unit value prices, p^0, where a unit value price of a product *i*, p_i^0, is defined in Equation 1:

$$\boldsymbol{p_i^0} = \frac{\sum_{m=1}^{12} p_i^{0,m}\, q_i^{0,m}}{\sum_{m=1}^{12} q_i^{0,m}} \quad (\mathbf{1})$$

In Equation 1, $p_i^{0,m}$ and $q_i^{0,m}$ denote a price and a quantity of a product *i* in a month *m* of a base year *0*. Altogether, Rothwell method for a month *m* of a comparison year *t* is defined in Equation 2 (ILO et al. 2004, p. 413):

$$\boldsymbol{P_R} = \frac{\sum_{i=1}^{I} p_i^{t,m}\, q_i^{0,m}}{\sum_{i=1}^{I} p_i^0 q_i^{0,m}} \quad (\mathbf{2})$$

In Equation 2, $p_i^{t,m}$ and $q_i^{0,m}$ denote a price of a product *i* in a month *m* of a comparison year *t* and a quantity of a product *i* in a month *m* of a base year *0*. Moreover, p_i^0 denotes a unit value price of a product *i* in a month *m* of a base year *0* defined in Equation 1.

Variable Weights Method

Variable Weights method uses changing (moving) weights, which vary from month to month. These weights change monthly in accordance with the changes in quantities observed during the same months of a base year. With this in mind, a weight of a product *i* in a month *m*, where *m = 1,2,…,12,* of a base year *0*, $s_i^{0,m}$, from a vector of changing (moving) weights, $s^{0,m}$, is defined in Equation 3:

$$\boldsymbol{s_i^{0,m}} = \frac{p_i^{0,m}\, q_i^{0,m}}{\sum_{i=1}^{I} p_i^{0,m}\, q_i^{0,m}} \quad (\mathbf{3})$$

In Equation 3, $p_i^{0,m}$ and $q_i^{0,m}$ denote a price and a quantity of a product *i* in a month *m* of a base year *0*. Altogether, Variable Weights method for a month *m* of a comparison year *t* is defined in Equation 4 (ILO et al. 2004):

$$\boldsymbol{P_{VW}} = \sum_{i=1}^{I} s_i^{0,m} \frac{p_i^{t,m}}{p_i^0} \quad (\mathbf{4})$$

In Equation 4, $p_i^{t,m}$ and $s_i^{0,m}$ denote a price of a product *i* in a month *m* of a comparison year *t* and a weight of a product *i* in a month *m* of a base year *0* defined in Equation 3. Moreover, p_i^0 denotes a unit value price of a product *i* in a month *m* of a base year *0* defined in Equation 1.

Fixed Weights Method

Fixed Weights method uses the same (fixed) weights during all of the months and imputes prices for missing products. As it is outlined in Section 2, these price imputations are based on an overall mean technique in this research. With this in mind, a yearly average fixed weight of a product *i* of a base year *0*, s_i^0, from a vector of yearly average fixed weights, s^0, is defined in Equation 5:

$$\boldsymbol{s_i^0} = \frac{\sum_{m=1}^{12} \boldsymbol{p_i^{0,m}\ q_i^{0,m}}}{\sum_{m=1}^{12} \sum_{i=1}^{I} \boldsymbol{p_i^{0,m}\ q_i^{0,m}}} \quad (\mathbf{5})$$

In Equation 5, $p_i^{0,m}$ and $q_i^{0,m}$ denote a price and a quantity of a product *i* in a month *m* of a base year *0*. Moreover, an average of matched products` price relatives necessary for price imputations is defined in Equation 6:

$$\boldsymbol{APR} = \frac{\sum_{i \in U^{m,m-1}} \boldsymbol{s_i^0} \frac{\boldsymbol{p_i^{t,m}}}{\boldsymbol{p_i^0}}}{\sum_{i \in U^{m,m-1}} \boldsymbol{s_i^0} \frac{\boldsymbol{p_i^{t,m-1}}}{\boldsymbol{p_i^0}}} \quad (\mathbf{6})$$

In Equation 6, $p_i^{t,m}$ and $p_i^{t,m-1}$ denote prices of a product *i* in months *m* and *m-1* of a comparison year *t* for matched products` set between months *m* and *m-1, $U^{m,m-1}$*. Moreover, p_i^0 and s_i^0 denote a unit value price and a yearly average fixed weight of a product *i* of a base year *0* defined in Equation 1 and in Equation 5. Price imputations for missing products are estimated from Equation 7:

$$\boldsymbol{p_i^{t,m}} = \boldsymbol{p_i^{t,m-1}} * \boldsymbol{APR} \quad (\mathbf{7})$$

In Equation 7, $p_i^{t,m}$ and $p_i^{t,m-1}$ denote prices of a product *i* (from unmatched products` set) in months *m* and *m-1* of a comparison year *t*. Moreover, APR denotes an average of matched products` price relatives defined in Equation 6. After the price imputations, Fixed Weights method for a month *m* of a comparison year *t* is defined altogether in Equation 8 (ILO et al. 2004):

$$\boldsymbol{P_{FW}} = \sum_{i=1}^{I} \boldsymbol{s_i^0} \frac{\boldsymbol{p_i^{t,m}}}{\boldsymbol{p_i^0}} \quad (\mathbf{8})$$

In Equation 8, $p_i^{t,m}$ and s_i^0 denote a price of a product *i* in a month *m* of a comparison year *t* and a yearly average fixed weight of a product *i* of a base year *0* defined in Equation 5. Moreover, p_i^0 denotes a unit value price of a product *i* in a month *m* of a base year *0* defined in Equation 1.

Arguably, an advantage of Fixed Weights method lies it its usage of only monthly prices and not of monthly quantities. This is different compared to the other 2 methods discussed in this section. A major disadvantage of this method lies in the necessity to make price imputations. This is again different compared to the other 2 methods discussed in this section. Moreover, a fixed weights structure of Fixed Weights method also poses a disadvantage. This is since this method by definition does not capture real monthly consumption patterns of consumers properly, which are rarely stable (ILO et al. 2020, Chapter 11). Likewise, one of the major disadvantages of Variable Weights and Rothwell methods` usage lies in a similar matter.

Quantities and weights used for these methods reflect fluctuations of a base year` months. This implies that these methods also do not capture real monthly consumption patterns of consumers properly. Another major disadvantage of these methods lies in their usage of only matched products between months m of a base year and of a comparison year. This implies that missing products are completely ignored. All of this suggests that traditional bilateral methods outlined in this section have disadvantages. This claim is further discussed in Section 6, while presenting and discussing empirical results.

4. Multilateral Methods

Gini-Eltetö-Köves-Szulc Method

Before proceeding to Gini-Eltetö-Köves-Szulc (hereinafter GEKS) method, it is important to describe Törnqvist method first (Törnqvist 1936). This is since Törnqvist method serves as a base method for GEKS calculations in this research. Törnqvist method uses a price vector, p^t, and an expenditure weights vector, s^t, for a matched products` set, $U_M^{o,t}$, with $t = 0,...,T$. With this in mind, Törnqvist method from a base month *0* to a comparison month *t* for a matched products` set, $U_M^{o,t}$, is defined in Equation 9:

$$\boldsymbol{P_T^{o,t}} = \prod_{\boldsymbol{i \in U_M^{o,t}}} \left(\frac{\boldsymbol{p_i^t}}{\boldsymbol{p_i^0}}\right)^{\left(\frac{\boldsymbol{s_i^0 + s_i^t}}{\boldsymbol{2}}\right)} \quad \textbf{(9)}$$

In Equation 9, p_i^0 and p_i^t denote prices of a product *i* in a base month *0* and in a comparison month *t*. Moreover, s_i^0 and s_i^t denote expenditure weights of a product *i* in a base month *0* and in a comparison month *t* estimated from $s_i^0 = \frac{p_i^0 q_i^0}{\sum_{j \in U_M^{o,t}} p_j^0 q_j^0}$ and from $s_i^t = \frac{p_i^t q_i^t}{\sum_{j \in U_M^{o,t}} p_j^t q_j^t}$.

Törnqvist method has several important properties. Arguably, one of the most important of them is that Törnqvist price indices are not transitive. This implies that Törnqvist price indices between 2 months depend on the choice of a base month (de Haan and Krsinich 2012, p. 4). The transitivity property is desirable since it suggests that the direct and the corresponding chained price indices are the same. If this property is not fulfilled, price indices might be subject to "chain drift". Formally, "chain drift" occurs if a chained price index, unlike its direct counterpart, is not equal to 1 when all prices revert back to their original base months values. "Chain drift" is usually caused by activities of sales and discounts which lead to stock keeping and it usually has a downward nature (Feenstra and Shapiro 2003, de Haan and van der Grient 2011, Diewert and Fox 2018).

To avoid "chain drift" while accounting for a high number of missing products, Ivancic, Diewert and Fox (2011) propose to use GEKS method (Gini 1931, Eltetö and Köves 1964, Szulc 1964). This method uses all possible matched products to calculate a price index as an unweighted geometric average of T+1 matched-model bilateral price indices P^{0l} and P^{lt} ratios, with l denoting a link month running through *t*, with *t = 0,...,T*. With this in mind, GEKS method from a base month *0* to a comparison month *t* is defined in Equation 10:

$$\boldsymbol{P_{GEKS}^{0,t}} = \prod_{\boldsymbol{l=0}}^{\boldsymbol{T}} \left(\boldsymbol{P^{0l}}/\boldsymbol{P^{tl}}\right)^{\left(\boldsymbol{1}/(\boldsymbol{T+1})\right)} = \prod_{\boldsymbol{l=0}}^{\boldsymbol{T}} \left(\boldsymbol{P^{0l}P^{lt}}\right)^{\left(\boldsymbol{1}/(\boldsymbol{T+1})\right)} \quad \textbf{(10)}$$

GEKS method has several important properties as well. First of all, this method passes Walsh`s multiperiod identity test (Walsh 1901, p. 379) (Walsh 1921, p. 540). This implies that GEKS price indices are “chain drift” free. Second of all, since all of the available months are taken as base months, they all make contributions to GEKS price indices. Third of all, since all of the possible bilateral product matches are considered, incorporation of new products is possible. Finally, strongly seasonal products might also make contributions to GEKS price indices if a comparison window is long enough to capture products` seasonal developments.

Geary-Khamis Method

One of the other alternative methods, which avoids “chain drift” while accounting for a high number of missing products as well, is Geary-Khamis (hereinafter GK) method (Chessa 2016). GK method uses unit value concept. It also uses quality adjustment factors, v_i, since aggregation of quantities is cumbersome due to their non-homogeneous nature. More specifically, quality adjustment factors make transformations of quantities to common units, $v_i q_i^t$, while also transforming prices to quality adjusted prices, p_i^t / v_i. These transformations lead to a quality adjusted unit value, $\tilde{p}^t$, in month *t* for a set of products, U_t, which is defined in Equation 11:

$$\tilde{\boldsymbol{p}}^{\boldsymbol{t}} = \frac{\sum_{i \in U_t} \boldsymbol{p}_i^t \boldsymbol{q}_i^t}{\sum_{i \in U_t} \boldsymbol{v}_i \boldsymbol{q}_i^t} \quad (\mathbf{11})$$

In Equation 11, p_i^t and q_i^t denote a price and a quantity of a product *i* in a month *t*. Moreover, v_i denotes quality adjustment factor of a product *i*. With this in mind, GK method from a base month *0* to a comparison month *t* is defined in Equation 12:

$$\boldsymbol{P}_{GK}^{0,t} = \frac{\tilde{\boldsymbol{p}}^t}{\tilde{\boldsymbol{p}}^0} = \frac{\sum_{i \in U_t} \boldsymbol{p}_i^t \boldsymbol{q}_i^t / \sum_{i \in U_0} \boldsymbol{p}_i^0 \boldsymbol{q}_i^0}{\sum_{i \in U_t} \boldsymbol{v}_i \boldsymbol{q}_i^t / \sum_{i \in U_0} \boldsymbol{v}_i \boldsymbol{q}_i^0} \quad (\mathbf{12})$$

In Equation 12, p_i^0 and p_i^t denote prices of a product *i* in a base month *0* and in a comparison month *t* as well as q_i^0 and q_i^t denote quantities of a product *i* in a base month *0* and in a comparison month *t*. Moreover, U_0 and U_t denote sets of products in a base month *0* and in a comparison month *t*. Quality adjustment factor, v_i, is defined in Equation 13:

$$\boldsymbol{v}_i = \frac{\sum_{z=0}^{T} \boldsymbol{q}_i^z \boldsymbol{p}_i^z / \boldsymbol{P}_{GK}^{0,z}}{\sum_{z=0}^{T} \boldsymbol{q}_i^z} \quad (\mathbf{13})$$

GK method uses quality adjustment factors, which are themselves used to calculate GK price indices. Therefore, Equation 12 and Equation 13 shall be solved simultaneously. This can be done by using iterative method.

Weighted Time Product Dummy Method

Another alternative method, which also avoids “chain drift” while accounting for a high number of missing products (Ivancic et al. 2011), is weighted time product dummy (hereinafter WTPD) method. WTPD method is an adaptation of the (unweighted) country product dummy method by Summers (1973). This adaptation, in context of price indices, is used by Aizcorbe, Corrado and Dones (2003), by Rao (2004), by Diewert (2005) and by many others. WTPD method is based on a weighted least squares regression, which is run on pooled data of all available months. In this regression, expenditure weights are the respective regression weights. These weight are estimated from $s_i^t = p_i^t q_i^t / \sum_{i=1}^{I} p_i^t q_i^t$. With this in mind, WTPD method with *N* products within *t* months, with *t = 0,…,T*, is defined in Equation 14:

$$\boldsymbol{ln\, p_i^t = \alpha + \sum_{t=1}^{T} \delta^t D_i^t + \sum_{i=1}^{N-1} \gamma_i D_i + \varepsilon_i^t} \ \ (\mathbf{14})$$

In Equation 14, time dummy variable, D_i^t, has a value of *1*, if observation relates to a month *t*, and has a value of *0* otherwise. Moreover, product dummy variable, D_i, has a value of *1*, if observation relates to a product *i*, and has a value of *0* otherwise. δ^t and γ_i coefficients denote time dummy and product dummy coefficients. Due to possible multicollinearity, an arbitrary product *N* is excluded from the regression. Conventionally, an estimated fixed effect of a product *i* is equal to *exp(*$\hat{\gamma}_i$*)* and an estimated WTPD price index, $P_{WTPD}^{0,t}$, is equal to *exp(*δ^t*)*.

Crucially, even if GEKS, GK and WTPD methods are all “chain drift” free, this does not mean that these methods are also free from the other biases. For instance, Greenlees and McClelland (2010) consider apparel products and claim that GEKS price indices could still be affected by a downward drift. With this in mind, de Haan and Krsinich (2012) suggest that this is due to a lack of explicit quality adjustment. To tackle this, they propose to substitute Törnqvist method with Imputation Törnqvist method for GEKS calculations to make price imputations for missing products.

Imputation Törnqvist GEKS Method

Imputation Törnqvist method from a base month *0* to a comparison month *t*, with *t = 0,…,T*, is defined in Equation 15:

$$\boldsymbol{P_{IT}^{o,t} = \prod_{i \in U_M^{o,t}} \left(\frac{p_i^t}{p_i^0}\right)^{\left(\frac{s_i^0 + s_i^t}{2}\right)} \prod_{i \in U_D^{o,t}} \left(\frac{\hat{p}_i^t}{p_i^0}\right)^{\left(\frac{s_i^0}{2}\right)} \prod_{i \in U_N^{o,t}} \left(\frac{p_i^t}{\hat{p}_i^0}\right)^{\left(\frac{s_i^t}{2}\right)}} \ \ (\mathbf{15})$$

In Equation 15, $U_M^{o,t}$ denotes a subset of matched products, which are available both during a base month *0* and a comparison month *t*. Moreover, $U_D^{o,t}$ denotes a subset of disappearing products, which are available in a base month *0* but not in a comparison month *t*. Similarly, $U_N^{o,t}$ denotes a subset of new products, which are available in a comparison month *t* but not in a base month *0*. Altogether, condition of $U^o \cup U^t = U_M^{o,t} \cup U_D^{o,t} \cup U_N^{o,t}$ is satisfied. Since some prices for a base month *0* and for a comparison month *t* are not available, $\hat{p}_i^t$ and $\hat{p}_i^0$ in Equation 15 denote price imputations. Crucially, there are several options how to make these price imputations, some of which are explored by de Haan and Krsinich (2012). They focused on price imputations, which are based either on time dummy hedonic or on WTPD methods. The former method uses hedonic regression, which requires products` quality characteristics, and produces quality adjusted price indices (de Haan and Daalmans 2019). The latter method does not require products` quality characteristics and does not produce quality adjusted price indices (de Haan and Hendriks 2013). Since scanner data does not usually offer products` quality characteristics, price imputations in this research are based on WTPD method.

To present Imputation Törnqvist GEKS (hereinafter ITGEKS) method, several steps need to be described. First of all, WTPD regression defined in Equation 14, is run on pooled data of all available months to obtain $\hat{p}_i^t$ and $\hat{p}_i^0$. These price imputations are estimated from $\hat{p}_i^t = exp(\hat{\alpha} + \delta^t + \hat{\gamma}_i)$ and from $\hat{p}_i^0 = exp(\hat{\alpha} + \hat{\gamma}_i)$ of WTPD regression coefficients. Second of all, all possible Imputation Törnqvist price indices are calculated in accordance with Equation 15. Finally, GEKS method is supplied with Imputation Törnqvist price indices to obtain ITGEKS price indices.

Logically, GEKS and ITGEKS price indices are the same if all products are available during all months. This is since no price imputations are made. However, when the number of missing products increases, price indices of these methods start to become different. Crucially, an option to make price imputations, which are based on WTPD regression run on pooled data of all available months, is somewhat biased. This is since products` fixed effects stay fixed over many months in WTPD regression and since WTPD regression deviates from bilateral comparison concept. However, since bilateral WTPD regression price imputations for ITGEKS calculations are inefficient (de Haan and Krsinich 2012), the above not previously employed option is used in this research.

5. Description of Seasonal Scanner Data Samples

The following seasonal scanner data samples are retrieved from real STATEC seasonal scanner data.

- Sample 1: 12 strongly seasonal products (products` seasons of availability do not change over comparison years)

- Sample 2: 6 strongly seasonal products (products` seasons of availability do not change over comparison years) and 6 strongly seasonal products (products` seasons of availability slightly change over comparison years)

- Sample 3: 4 strongly seasonal products (products` seasons of availability do not change over comparison years), 4 strongly seasonal products (products` seasons of availability slightly change over comparison years) and 4 random strongly seasonal products (products` seasons of availability are not pronounced over comparison years)

- Sample 4: 3 strongly seasonal products (products` seasons of availability do not change over comparison years), 3 strongly seasonal products (products` seasons of availability slightly change over comparison years), 3 random strongly seasonal products (products` seasons of availability are not pronounced over comparison years) and 3 weakly seasonal products (available over comparison years)

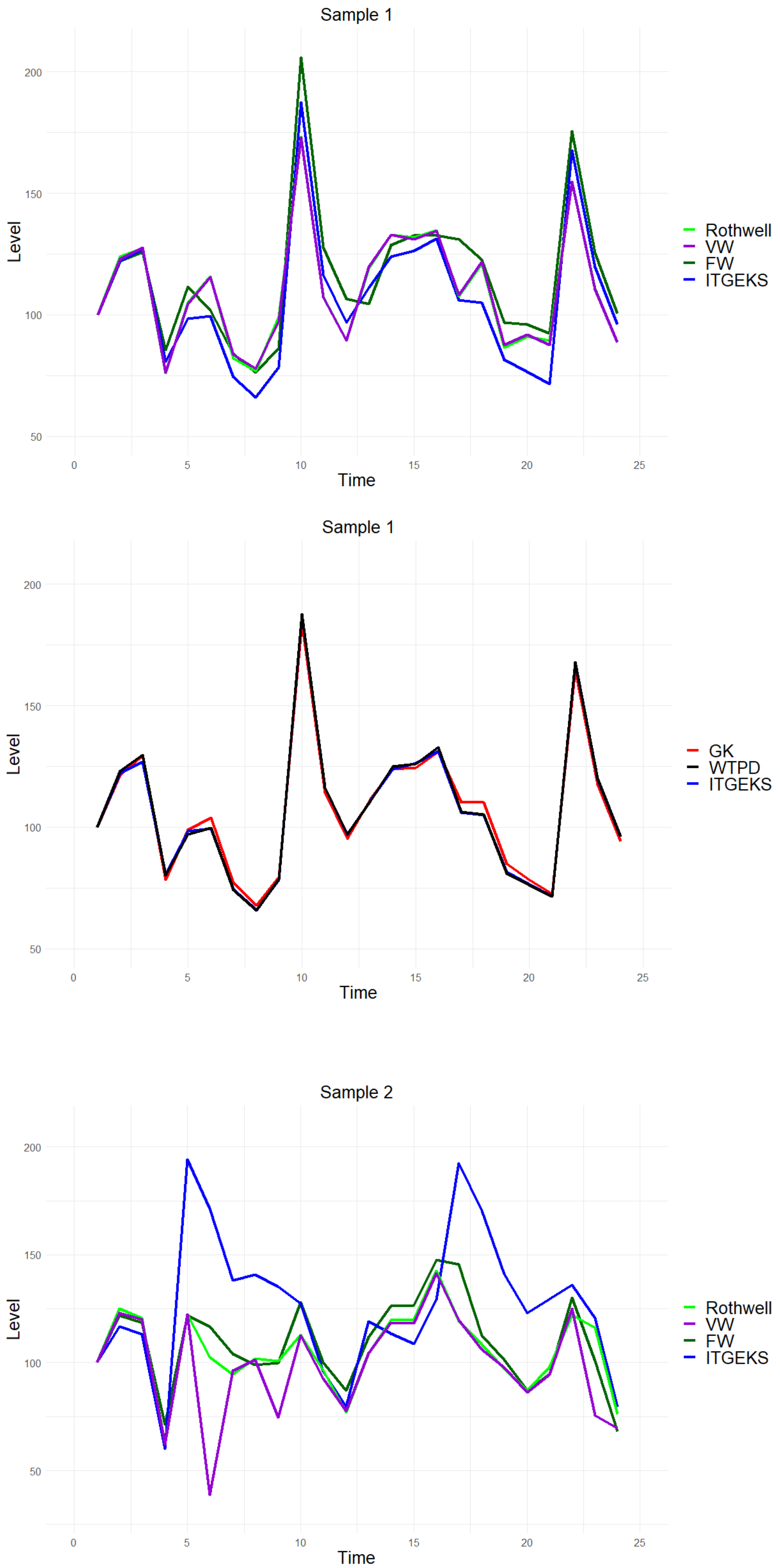
The above seasonal scanner data samples are constructed to gradually increase the complexity level of the comparison. On one hand, Sample 1 presents a somewhat ideal basket of only having strongly seasonal products. Such a basket is usually far from being a real seasonal scanner data basket. On the other hand, Sample 4 presents a basket of having both strongly and weakly seasonal products. Such a basket, on the contrary, normally corresponds to a real seasonal scanner data basket.

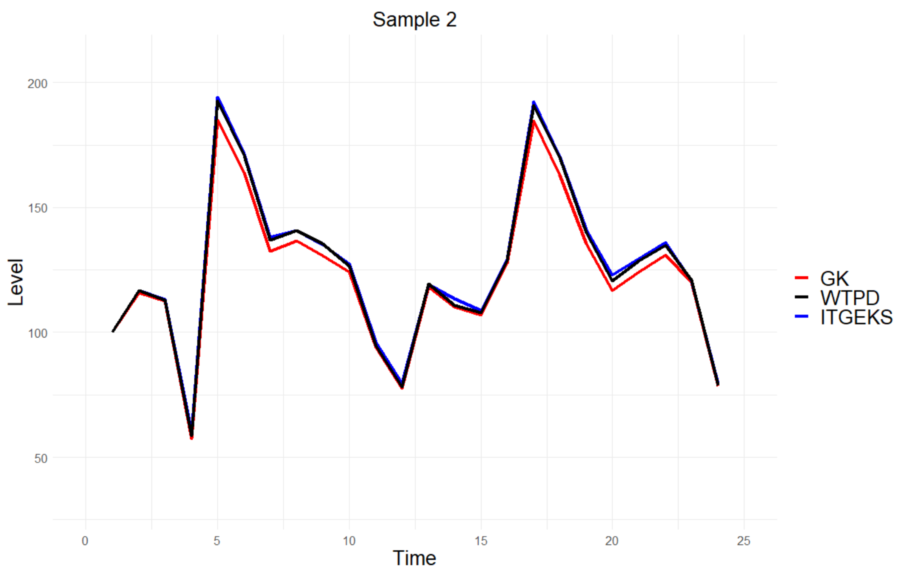
6. Empirical Results

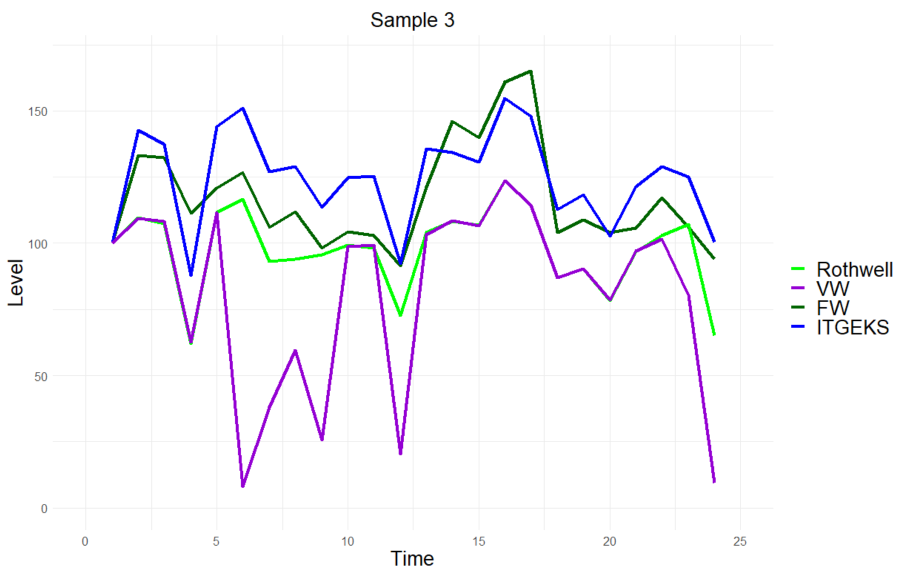
After the description of traditional bilateral methods and the most common multilateral methods as well as of 4 seasonal scanner data samples, the first empirical results are provided in Figure 1:

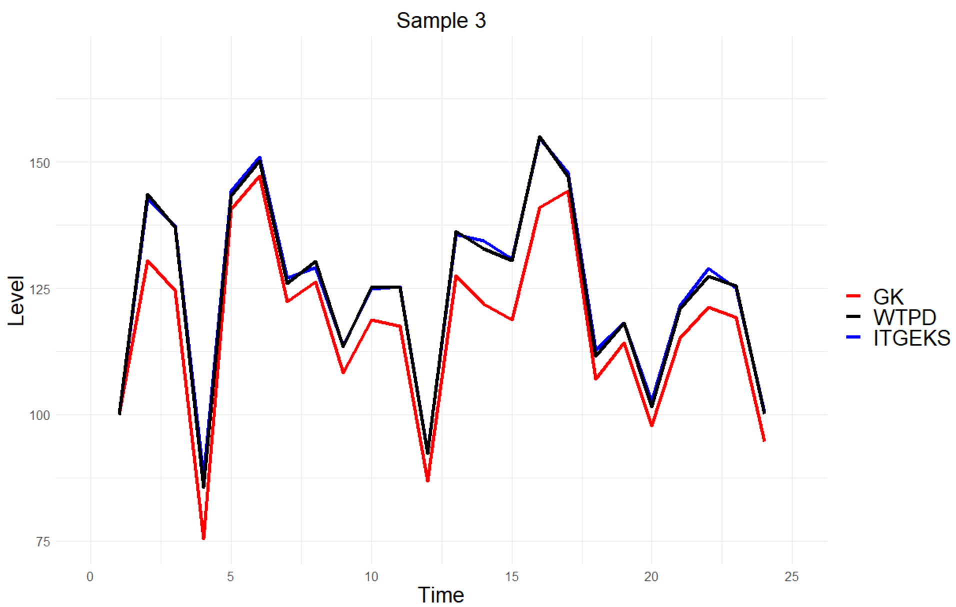
In Figure 1, price indices of Fixed Weights (hereinafter FW), GEKS, GK, ITGEKS, Rothwell, Variable Weights (hereinafter VW) and WTPD methods are displayed. These price indices are obtained from 36 months of seasonal scanner data. In Figure 1, ITGEKS method serves as a benchmark method due to its importance in this research, which is discussed below.

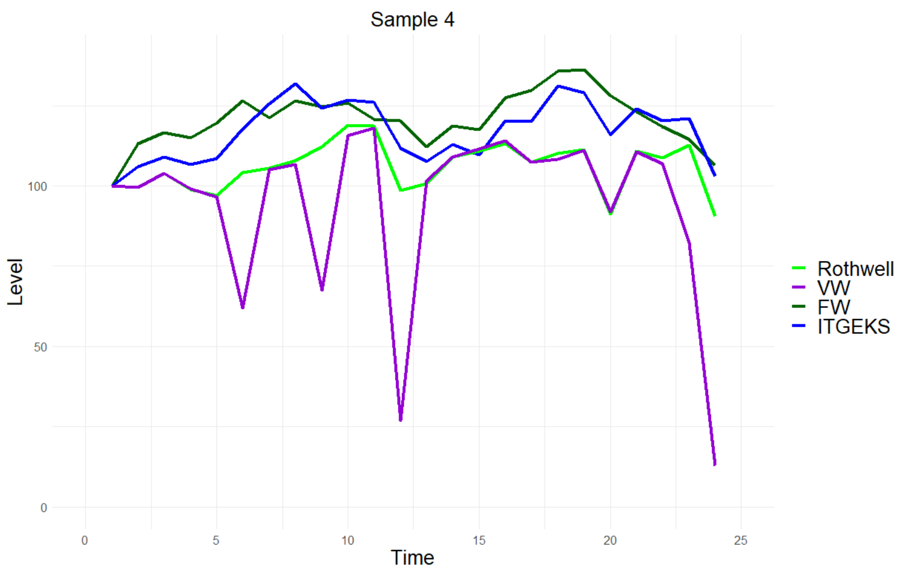
Figure 1: Price Indices of the Selected Bilateral and Multilateral Methods

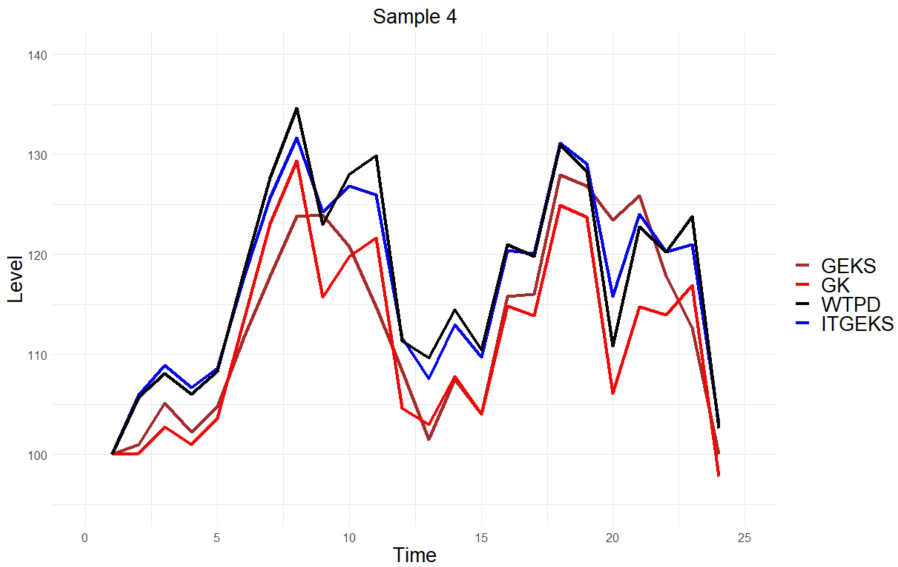



Sample 2
Level
Time
GK
WTPD
ITGEKS


Sample 3
Level
Time
Rothwell
VW
FW
ITGEKS


Sample 3
Level
Time
GK
WTPD
ITGEKS





It is noteworthy that FW, Rothwell and VW methods all use prices and quantities of a base year. This implies that price indices for a base year cannot be calculated using these methods. With this in mind, price indices of all selected methods for the last 24 months are provided. These price indices are adjusted to be equal to 100 in their first months. Crucially, price indices displayed in Figure 1 do not include GEKS price indices for Samples 1, 2 and 3. This is since seasonal scanner data of these samples is fully strongly seasonal and sometimes contains no bilateral product matches for GEKS calculations. Therefore, proper GEKS calculations for these samples are impossible. This is one of the major disadvantages of GEKS method, which is discussed below.

As it can be seen from Figure 1, price indices of all selected methods are similar for Sample 1. However, they are becoming different when the other, more realistic, seasonal scanner data samples are considered. These differences can be partly explained from traditional bilateral methods` standpoint as follows. First of all, downward drift in Rothwell and VW price indices of Samples 2, 3 and 4, is mainly caused by changing products` seasons of availability over years. Second of all, even though FW price indices do not show any downward drift, these price indices are still not very accurate since they do not capture real monthly consumption patterns of consumers properly. It is noteworthy that VW price indices break off in some months of Samples 2, 3 and 4 since there are positive weights but no prices for some products. All this implies that traditional bilateral methods should not be used for seasonal products. Crucially, multilateral methods do not have the above disadvantages, which are avoided by their structures. Therefore, the focus of the rest of this research is set on comparison and discussion of multilateral methods.

Comparison and Discussion of GEKS, GK and WTPD Methods

The decision which multilateral method should be preferred in practice is not apparent. This is since price indices of multilateral methods might be different when seasonal products are considered. This can be seen from price indices of Figure 1. These differences can be partly explained by different structures and by different expenditures` treatment of multilateral methods. However, these facts do not simplify the above decision. Therefore, to possibly make this decision, this research compares and discusses several important properties of the most common multilateral methods.

Comparing GEKS, GK and WTPD methods, it can be claimed that GEKS price indices are less sensitive to a usage of different splices rather than GK and WTPD price indices. Splices are created to ensure continuity of price indices while accounting for non-revision of the previously published price indices. The 4 traditional splices are movement (van der Grient and de Haan 2011), window (Krsinich 2016), half (de Haan 2015) and mean (Diewert and Fox 2017) splices. Moreover, there are also splices, which are based on a usage of published price indices - window splice on published price indices (hereinafter WISP), half splice on published price indices (hereinafter HASP) and mean splice on published price indices (hereinafter MESP)[2]. Finally, there is fixed base based splice as well - fixed base moving window (hereinafter FBMW) splice (Lamboray 2017). These splices are all based on a rolling window approach, which shifts window (usually of 13 or 25 months for seasonal products) by 1 month and splices new price indices on existing price indices. Crucially, splices make GEKS, GK and WTPD price indices only approximately “chain drift” free. This is since there is always some degree of chaining involved when splices are used. With this in mind, the above claim suggests that spliced GEKS price indices are closer to be “chain drift” free comparing to their spliced GK and WTPD counterparts.

To show how the splices exactly work and to reconfirm the claim of the above paragraph, it is important to formally describe the splices first. With this in mind, the splices are defined in Equations 16, 17, 18, 19 and 20 below. After such definition, the spliced GEKS, GK, ITGEKS and WTPD price indices are provided in Figure 2.

Movement splice calculates a price index for a new month *t* by chaining the last month month-to-month price index of the shifted window to a price index of the previous month calculated over the previous window.

$$\boldsymbol{P_{MS}^{0,t} = P_{MS}^{0,t-1} P_{t-T+1,t}^{t-1,t}} \quad (\mathbf{16})$$

Window splice calculates a price index for a new month *t* by chaining price indices to a price index calculated 24 months ago over the previous window if the window length is 25 months.

$$\boldsymbol{P_{WS}^{0,t} = P_{0,T}^{0,1} P_{1,T+1}^{1,2} \cdots P_{t-T,t}^{t-T+1,t}} \quad (\mathbf{17})$$

[2] Crucially, movement splice directly uses published price indices.

Half splice calculates a price index for a new month t by chaining in the middle of the window length. More specifically, half splice occurs at $t = \frac{T+1}{2}$ when T is odd and at $t = \frac{T}{2}$ when T is even. If window length is 25 months, splice occurs at 13th month of the window.

$$\boldsymbol{P_{HS}^{0,t} = P_{HS}^{0,t-1} \frac{P_{t-T+1,t}^{t-\frac{T+1}{2}+1,t}}{P_{t-T,t-1}^{t-\frac{T+1}{2}+1,t-1}}} \quad (\mathbf{18})$$

Mean splice calculates a price index for a new month t by using a geometric mean of all possible spliced months` options.

$$\boldsymbol{P_{MS}^{0,t} = P_{MS}^{0,t-1} \prod_{l=t-T+1}^{t-1} \left(\frac{P_{t-T+1,t}^{l,t}}{P_{t-T,t-1}^{l,t-1}}\right)^{\frac{1}{T-1}}} \quad (\mathbf{19})$$

Splices on published price indices are similar to traditional splices, with an exception that published price indices and not recalculated price indices are used. An advantage of these splices lies in avoidance of "base level effect". Moreover, another major advantage, which is only attributed to a usage of HASP on 25 months` window, lies in consistency of the derived annual and published annual rates. This advantage makes HASP on 25 months` window a good option for producing official price indices.

FBMW splice calculates a price index for a new month t by comparing the last month of the window to the fixed base, which is usually a December month of the previous year.

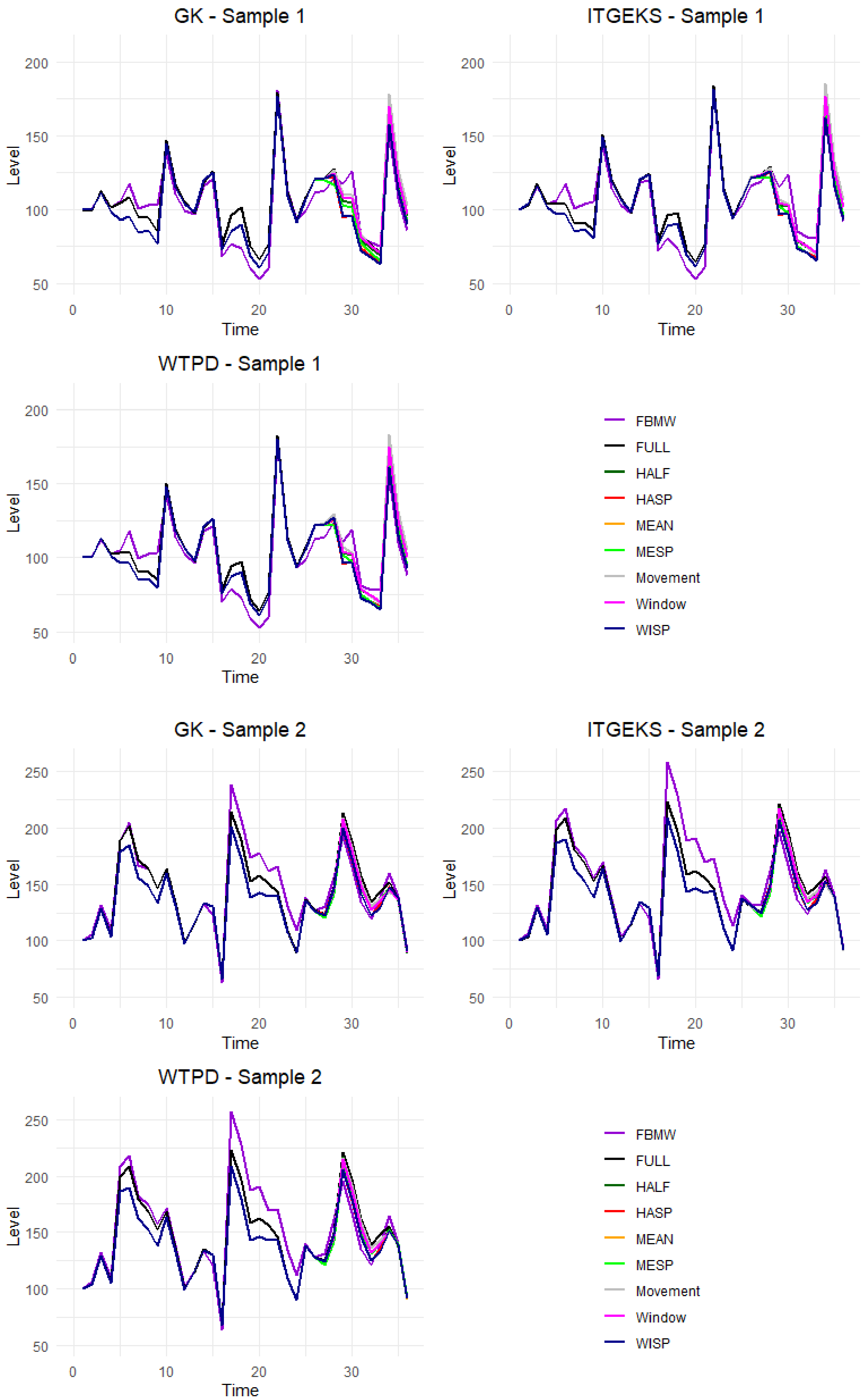
$$\boldsymbol{P_{FBMW}^{0,t} = P_{b-T,b}^{b-T,b} P_{t-T,t}^{b,t}} \quad (\mathbf{20})$$

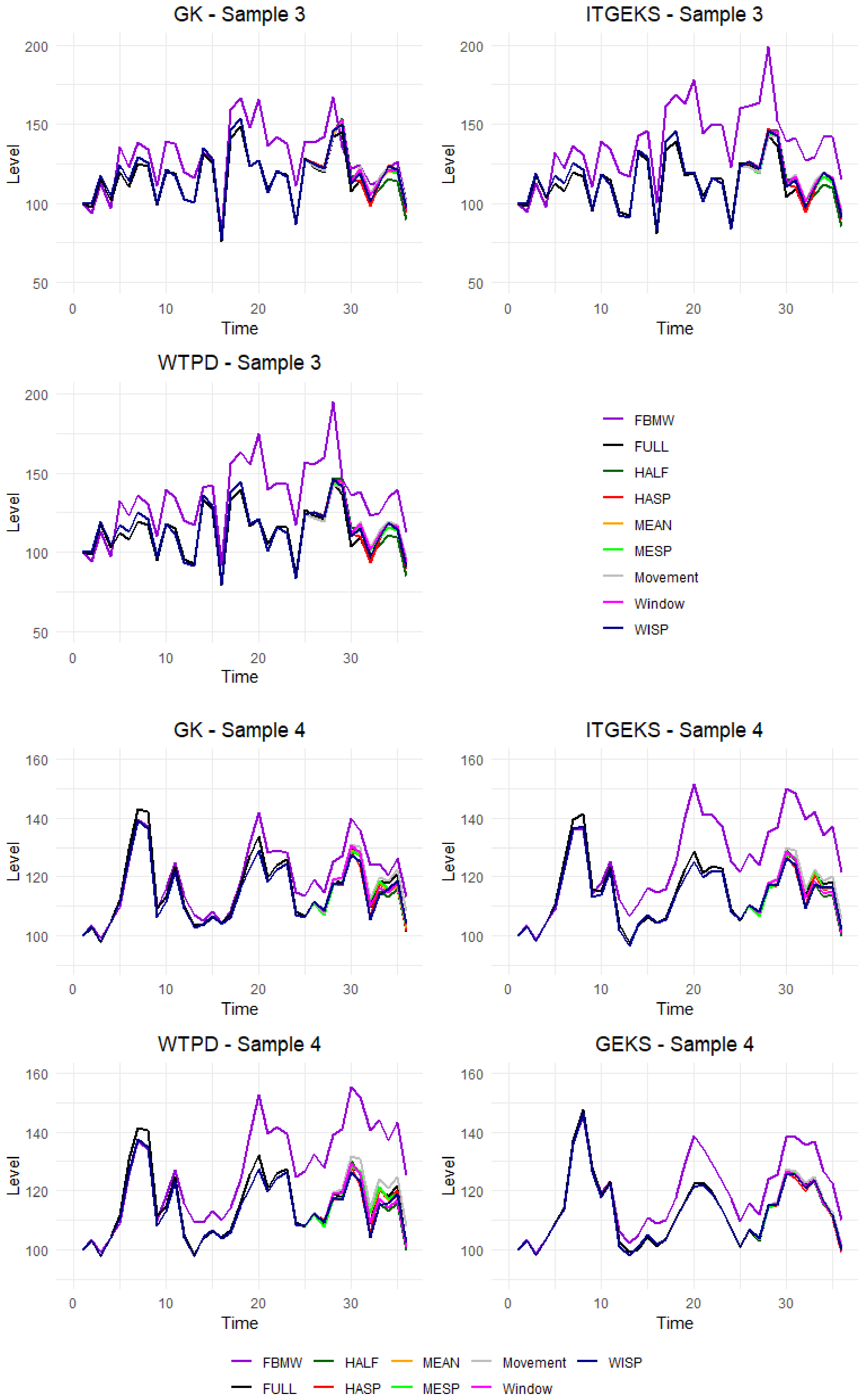
In Figure 2, the spliced price indices of the most common multilateral methods are displayed. Moreover, Figure 2 also provides full window (FULL) price indices, which are seen as benchmark price indices since they are "chain drift" free. Window length for all splices, except FBMW splice (13 months), is 25 months. This is since even though there is no consensus regarding an optimal window length, it seems that there is a convergence to prefer larger to smaller window lengths, especially if strongly seasonal products are considered (Chessa 2019). Crucially, price indices displayed in Figure 2 do not include GEKS price indices for Samples 1, 2 and 3 due to the same reason outlined in the previous subsection.

As it can be seen from Figure 2 and especially from price indices of Sample 4, the claim that GEKS price indices are less sensitive to a usage of different splices rather than GK and WTPD price indices is indeed fully justified. More specifically, the differences between GEKS benchmark and the spliced GEKS price indices lie in a small range from -2.5 to 1 percent. On the contrary, the differences of similar comparisons for GK and WTPD methods lie in a significant range from -7.5 to 5 percent. It can also be seen from Figure 2 that FBMW price indices significantly deviate from benchmark price indices. This shows that FBMW splice should probably not be preferred for seasonal products.

Comparing GEKS, GK and WTPD methods further, it can be also claimed that WTPD method is an approximately additive method (Diewert and Fox 2018). The additivity property suggests that a method is in line with consumers, who buy products while maximizing a linear utility function.

Figure 2: The Spliced Price Indices of GEKS, GK, ITGEKS and WTPD Methods




GK - Sample 3
ITGEKS - Sample 3
WTPD - Sample 3
GK - Sample 4
ITGEKS - Sample 4
WTPD - Sample 4
GEKS - Sample 4
Level
Time
200
150
100
50
160
140
120
0
10
20
30
FBMW
FULL
HALF
HASP
MEAN
MESP
Movement
Window
WISP

This implies that it is implicitly assumed that products are perfectly substitutable after quality adjustments. Crucially, any additive method, and hence WTPD method, is not generally consistent with economic approach to index number theory, which assumes that consumers` utility functions are generally not linear. However, these, more realistic, utility functions can be approximated by GEKS method reasonably well.

Comparing GEKS, GK and WTPD methods even further, axiomatic approach to index number theory should be also considered. Since traditional tests are created for a fixed products` universe, this research is focused on tests proposed by Zhang, Johansen and Nygaard (Zhang et al. 2018) at first. These tests are created for a dynamic products` universe accounting for missing products. While introducing identity, fixed basket, upper bound, lower bound and responsiveness tests in their research, it is perhaps the very last test, which deserves a specific attention. This is since responsiveness test checks of whether a multilateral method is sensitive to imputed prices or not. Crucially, it can be claimed that GEKS method passes and GK as well as WTPD methods fail this test (Zhang et al. 2018). This implies that GK and WTPD price indices are subject to new product bias (Boskin et al. 1996, Nordhaus 1997, Diewert 1998).

Summing up, GEKS method rather than GK and WTPD methods is less sensitive to a usage of different splices, is generally consistent with economic approach to index number theory and is responsive to price imputations. These advantages undoubtedly make GEKS method a better option rather than GK and WTPD methods to be used for seasonal products. This is since seasonal products usually experience high volatilities in prices and quantities, do not usually follow linear utility preferences and are usually missing. However, as it is outlined above, if only strongly seasonal products are considered and if there are no bilateral product matches, proper GEKS calculations are impossible. Therefore, the focus of the next subsection is set on comparison and discussion of GEKS and ITGEKS methods.

Comparison and Discussion of GEKS and ITGEKS Methods

As it is outlined in the previous subsection, GEKS method can be considered as a somewhat preferred method for seasonal products. However, the problem of no bilateral product matches, when only strongly seasonal products are considered, still remains. Crucially, ITGEKS method might be seen as a solution to this problem. This is since ITGEKS method ensures that the problem of no bilateral product matches is avoided.

If the spliced price indices are revisited, the claim that ITGEKS price indices are less sensitive to a usage of different splices rather than GK and WTPD price indices can be made and justified. More specifically, the differences between ITGEKS benchmark and the spliced ITGEKS price indices lie in a moderate range from -4 to 2.5 percent. On the contrary, the differences of similar comparisons for GK and WTPD methods lie in a significant range from -7.5 to 5 percent. However, the claim that ITGEKS price indices are more sensitive to a usage of different splices rather than GEKS price indices can also be made and justified. More specifically, the differences between GEKS benchmark and the spliced GEKS price indices lie in a small range from -2.5 to 1 percent.

If economic approach to index number theory is revisited, it can be claimed that ITGEKS but not GEKS method makes modelling of reservation (imputed) prices (Hicks 1940) for strongly seasonal products possible. If axiomatic approach to index number theory is revisited, a connection between ITGEKS and GEKS methods should be discussed. Since GEKS and not GK or WTPD method can incorporate effects of missing products, ITGEKS method can be seen as a method, which makes such an incorporation possible. Crucially, as it is shown in the

Appendix, it can be also claimed that ITGEKS method passes and fails the very same axiomatic tests, which are created for a fixed products` universe, as GEKS method. Finally, this research proposes several new tests, which are failed by GEKS method and are passed by ITGEKS method:

1) Responsiveness to isolated products test:

This test asks that if products are available only during 1 month in the window of *T* months, price level functions respond to changes in prices and/or changes in quantities of these products. This implies that price level functions are not constant if prices and/or quantities of isolated products change[3].

2) Strong isolated products test:

This test asks that if there are only isolated products in a sample, which are available only during 1 month in the window of *T* months, price level functions can still be calculated.

3) Weak isolated products test:

This test asks that if there are months in the window of *T* months, during which only isolated products are available, price level functions can still be calculated.

4) Convergence test:

This test asks that if the number of bilateral product matches in the window of *T* months decreases, price level functions do not converge to a certain number.

GEKS method fails and ITGEKS method passes the above tests due to the following reasons. GEKS method fails the first test since this method requires products to be available at least during 2 months in the window of *T* months to make their contributions to price indices. This requirement is not relevant for ITGEKS method since price imputations ensure that isolated products make their contributions to price indices. Crucially, this all implies that GEKS but not ITGEKS price indices are also subject to new product bias. Moreover, GEKS method fails the second and the third tests since this method breaks off if there are no bilateral product matches during all or some months. This requirement is not relevant for ITGEKS method since price imputations ensure that price indices can be calculated. Finally, GEKS method fails the fourth test since if the number of bilateral product matches decreases, GEKS price indices converge to a number, which has to be inserted instead of empty Törnqvist price indices (GEKS matrix cells) to calculate GEKS price indices. This is not relevant for ITGEKS method since price imputations ensure that Imputation Törnqvist price indices (ITGEKS matrix cells) are not the same.

7. Conclusion

It can be concluded from empirical results of this research that all traditional bilateral methods have disadvantages and should not be used for seasonal products. Moreover, it can be also concluded that GEKS method rather than GK or WTPD methods can be considered as a somewhat preferred method for seasonal products if proper GEKS calculations are possible. If proper GEKS calculations are impossible, ITGEKS method might be seen as an alternative to GEKS method.

[3] This test is proposed by Claude Lamboray. The authors extend this test`s definition for it to reflect not only on prices but also on quantity changes.

List of References:

ABS (2017). Making Greater Use of Transactions Data to Compile the Consumer Price Index. Information paper ABS.

Aizcorbe, A., C. Corrado and M. Doms (2003), When Do Matched-Model and Hedonic Techniques Yield Similar Price Measures? Working Paper no. 2003-14, Federal Reserve Bank of San Francisco.

Balk, B., (2001) Aggregation Methods in International Comparisons: What Have We Learned?

Bean, L. H., and Stine, O. C. (1924). Four types of index numbers of farm prices. Journal of the American Statistical Association, 19(145), 30-35.

Boskin, M. J. (1996). Toward a more accurate measure of the cost of living. Advisory Commission to Study the Consumer Price Index.

Chessa, A. (2016) A new methodology for processing scanner data in the Dutch CPI. - Eurostat review of National Accounts and Macroeconomic Indicators, 1/2016, 49-69.

Chessa, A. (2019) A comparison of index extension methods for multilateral methods.

de Haan, J., and Daalmans, J. (2019). Scanner Data in the CPI: The Imputation CCDI Index Revisited.

de Haan, J., and Hendriks, R. (2013, November). Online data, fixed effects and the construction of high-frequency price indexes. In *Economic Measurement Group Workshop* (pp. 28-29).

de Haan, J., and Krsinich, F. (2012). Scanner data and the treatment of quality change in rolling year GEKS price indexes. In EMG Workshop.

de Haan, J. (2015). A framework for large scale use of scanner data in the Dutch CPI. In Report from Ottawa Group 14th meeting, International Working Group on Price Indices, Tokyo (Japan), May, 2015. Ottawa Group.

Diewert, E. (1998). Index number issues in the consumer price index. Journal of Economic Perspectives, 12(1), 47-58.

Diewert, E. (1999). Axiomatic and Economic Approaches to International Comparisons. International and Interarea Comparisons of Income, Output, and Prices, Studies in Income and Wealth, Vol. 61, Chicago, University of Chicago Press.

Diewert, E. (2005). Weighted country product dummy variable regressions and index number formulae. Review of Income and Wealth, 51(4), 561-570.

Diewert, E. (2014). Recent Developments in Consumer Price Index Theory. Index number theory and measurement economics course material.

Diewert, W. E., and Fox, K. J. (2018). Substitution bias in multilateral methods for CPI construction using scanner data. UNSW Business School Research Paper, (2018-13).

Eltetö, O. and P. Köves (1964), On a Problem of Index Number Computation Relating to International Comparisons, Statisztikai Szemle 42, 507-518 (in Hungarian).

Feenstra, R. C., and Shapiro, M. D. (2003). High-frequency substitution and the measurement of price indexes. In Scanner Data and Price Indexes (pp. 123-150). University of Chicago Press.

Gini, C. (1931). On the circular test of index numbers. Metron, 9(9), 3-24.

Greenlees, J., and McClelland, R. (2010). Superlative and regression-based consumer price indexes for apparel using US scanner data. In Conference of the International Association for Research in Income and Wealth, St. Gallen, Switzerland.

Hicks, J. R. (1940). The valuation of the social income. *Economica*, *7*(26), 105-124.

ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2004), Consumer Price Index Manual: Theory and Practice. ILO Publications, Geneva.

ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2020), Consumer Price Index Manual: Theory and Practice. ILO Publications, Geneva.

Ivancic, L., Diewert, W. E., and Fox, K. J. (2011). Scanner data, time aggregation and the construction of price indexes. Journal of Econometrics, 161(1), 24-35.

Krsinich, F. (2016). The FEWS index: Fixed effects with a window splice.

Lamboray, C. (2017). The Geary Khamis index and the Lehr index: how much do they differ.

Nordhaus, W. D. (1996). Do real-output and real-wage measures capture reality? The history of lighting suggests not. In The economics of new goods (pp. 27-70). University of Chicago Press.

Rao, D. P. (2005). On the equivalence of Weighted Country Product Dummy (CPD) method and the Rao-system for multilateral price comparisons. Review of Income and Wealth, 51(4), 571-580

Rothwell, D. P. (1958). Use of varying seasonal weights in price index construction. Journal of the American Statistical Association, 53(281), 66-77.

Summers, R. (1973), International Price Comparisons Based Upon Incomplete Data,

Szulc, B. (1964), Indices for Multiregional Comparisons, Przeglad Statystycny 3, 239-254 (in Polish).

Törnqvist, L. (1936). The Bank of Finland's consumption price index.

van der Grient, H. A., and de Haan, J. (2011, March). Scanner Data Price Indexes: The ‘Dutch Method” versus Rolling Year GEKS”. In 12th Meeting of the Ottawa Group, May (pp. 4-6).

Walsh, C. M. (1901). The measurement of general exchange-value (Vol. 25). Macmillan.

Walsh, C. M. (1921). The best form of index number: Discussion. Quarterly Publications of the American Statistical Association, 17(133), 537-544.

Zhang, L. C., Johansen, I., and Nygaard, R. (2018). Tests for price indices in a dynamic item universe. arXiv preprint arXiv:1808.08995.

Appendix:

The following Appendix discusses axiomatic approach to index number theory for GEKS, GK, ITGEKS and WTPD methods to show which tests are passed by and are failed by ITGEKS method. To start with, Australian Bureau of Statistics (ABS 2017) using Diewert (1999) and Balk (2001) focuses on the tests, which are passed by and are failed by GEKS, GK and WTPD methods for a fixed products` universe. As it is shown below, these tests` results make it possible to make conclusions regarding ITGEKS method. With this in mind, the tests` results for GEKS, GK, ITGEKS and WTPD methods are provided in Table 1:

		GEKS	**GK**	**ITGEKS**	**WTPD**
Test 1	Transitivity test	YES	YES	YES	YES
Test 2	Identity test	NO	NO	NO	NO
Test 3	Multiperiod identity test	YES	YES	YES	YES
Test 4	Continuity, positivity and normalization test	YES	YES	YES	YES
Test 5	Proportional prices test	YES	YES	YES	YES
Test 6	Homogeneity in quantities test	YES	NO	YES	YES
Test 7	Homogeneity in prices test	YES	YES	YES	YES
Test 8	Commensurability test	YES	YES	YES	YES
Test 9	Symmetry in the treatment of time periods test	YES	YES	YES	YES
Test 10	Symmetry in the treatment of time products test	YES	YES	YES	YES
Test 11	Basket test	NO	YES	NO	NO
Test 12	Responsiveness to imputed prices test	YES	NO	YES	NO

Using the tests` results for GEKS and WTPD methods, the above tests` results for ITGEKS method, with an assumption that the same products are available in a base month and in a comparison month, are derived as follows. Crucially, for ITGEKS method to pass or to fail a test, it is sufficient to show that WTPD method passes or fails it. This is since after WTPD price imputations, ITGEKS method transforms into GEKS method. It is noteworthy that even if the above tests` results for WTPD method are for a fixed products` universe, they can still be used for a dynamic products` universe. This is since WTPD price indices on samples with some products with strictly positive prices and 0 quantities (a fixed products` universe) and WTPD price indices on samples with the same but missing products (a dynamic products` universe) are the same.

To formally postulate the above tests, it is assumed that prices and quantities vectors $p^t = (p_1^t, \dots, p_n^t)$ and $q^t = (q_1^t, \dots, q_n^t)$ are available for each month t and that $P^{0,t}(p^0, \dots, p^T, q^0, \dots, q^T) = P^{0,t}$ presents a multilateral method as a function of prices and quantities vectors from 0 to T months.

Test 1 – Transitivity test

Definition - transitivity test postulates that:

$$P^{0,t2} = P^{0,t1} * P^{t1,t2}$$

Proof - ITGEKS method passes transitivity test as follows:

for $P^{0,t2}_{ITGEKS} = P^{0,t1}_{ITGEKS} * P^{t1,t2}_{ITGEKS}$ to hold, it is sufficient to show that $\prod_{l=0}^{T}\left(P^{l,t1}_{IT}P^{t1,l}_{IT}\right)^{\left(1/(T+1)\right)} = 1$ holds.

This holds since $U^{l,t1}_{M} = U^{t1,l}_{M}$, $U^{l,t1}_{D} = U^{t1,l}_{N}$ and $U^{l,t1}_{N} = U^{t1,l}_{D}$ as follows:

$$\prod_{l=0}^{T}\left(P^{l,t1}_{IT}P^{t1,l}_{IT}\right)^{\left(1/(T+1)\right)}$$

$$= \prod_{l=0}^{T}\left(\prod_{i\in U^{l,t1}_{M}}\left(\frac{p^{t1}_{i}}{p^{l}_{i}}\right)^{\left(\frac{s^{l}_{i}+s^{t1}_{i}}{2}\right)}\prod_{i\in U^{l,t1}_{D}}\left(\frac{\hat{p}^{t1}_{i}}{p^{l}_{i}}\right)^{\left(\frac{s^{l}_{i}}{2}\right)}\prod_{i\in U^{l,t1}_{N}}\left(\frac{p^{t1}_{i}}{\hat{p}^{l}_{i}}\right)^{\left(\frac{s^{t1}_{i}}{2}\right)}\prod_{i\in U^{t1,l}_{M}}\left(\frac{p^{l}_{i}}{p^{t1}_{i}}\right)^{\left(\frac{s^{t1}_{i}+s^{l}_{i}}{2}\right)}\prod_{i\in U^{t1,l}_{D}}\left(\frac{\hat{p}^{l}_{i}}{p^{t1}_{i}}\right)^{\left(\frac{s^{t1}_{i}}{2}\right)}\prod_{i\in U^{t1,l}_{N}}\left(\frac{p^{l}_{i}}{\hat{p}^{t1}_{i}}\right)^{\left(\frac{s^{l}_{i}}{2}\right)}\right)^{\left(1/(T+1)\right)}$$

$$= \prod_{l=0}^{T}\left(\prod_{i\in U^{l,t1}_{M}}(1)^{\left(\frac{s^{l}_{i}+s^{t1}_{i}}{2}\right)}\prod_{i\in U^{l,t1}_{D}}(1)^{\left(\frac{s^{l}_{i}}{2}\right)}\prod_{i\in U^{l,t1}_{N}}(1)^{\left(\frac{s^{t1}_{i}}{2}\right)}\right)^{\left(1/(T+1)\right)} = 1$$

Test 3 – Multiperiod identity test

Definition - multiperiod identity test postulates that:

if $p^{T}_{i} = p^{0}_{i}$ and if $q^{T}_{i} = q^{0}_{i}$ for all $i = 1, \dots, n$,
then $P^{0,1} * P^{1,2} * \dots * P^{T-1,T} = 1$

Proof - ITGEKS method passes multiperiod identity test as follows:

for $P^{0,1}_{ITGEKS} * P^{1,2}_{ITGEKS} * \dots * P^{T-1,T}_{ITGEKS} = 1$,
when $p^{0}_{i} = p^{T}_{i} = p_{i}$ and $q^{0}_{i} = q^{T}_{i} = q_{i}$ for all $i = 1, \dots, n$, to hold,
it is sufficient to show that $P^{0,T}_{ITGEKS} = 1$ holds
since ITGEKS method passes transitivity test.

This holds since $U^{0,l}_{M} = U^{l,T}_{M}$, $U^{0,l}_{D} = U^{l,T}_{N}$, $U^{0,l}_{N} = U^{l,T}_{D}$, $\hat{p}^{0}_{i} = \hat{p}^{T}_{i} = \hat{p}_{i}$ and $s^{0}_{i} = s^{T}_{i} = s_{i}$ as follows:

$$P^{0,T}_{ITGEKS}$$

$$= \prod_{l=0}^{T}\left(\prod_{i\in U^{0,l}_{M}}\left(\frac{p^{l}_{i}}{p_{i}}\right)^{\left(\frac{s_{i}+s^{l}_{i}}{2}\right)}\prod_{i\in U^{0,l}_{D}}\left(\frac{\hat{p}^{l}_{i}}{p_{i}}\right)^{\left(\frac{s_{i}}{2}\right)}\prod_{i\in U^{0,l}_{N}}\left(\frac{p^{l}_{i}}{\hat{p}_{i}}\right)^{\left(\frac{s^{l}_{i}}{2}\right)}\prod_{i\in U^{l,T}_{M}}\left(\frac{p_{i}}{p^{l}_{i}}\right)^{\left(\frac{s^{l}_{i}+s_{i}}{2}\right)}\prod_{i\in U^{l,T}_{D}}\left(\frac{\hat{p}_{i}}{p^{l}_{i}}\right)^{\left(\frac{s^{l}_{i}}{2}\right)}\prod_{i\in U^{l,T}_{N}}\left(\frac{p_{i}}{\hat{p}^{l}_{i}}\right)^{\left(\frac{s_{i}}{2}\right)}\right)^{\left(1/(T+1)\right)}$$

$$= \prod_{l=0}^{T}\left(\prod_{i\in U_M^{0,l}}(1)^{\left(\frac{s_i+s_i^l}{2}\right)}\prod_{i\in U_D^{0,l}}(1)^{\left(\frac{s_i}{2}\right)}\prod_{i\in U_N^{0,l}}(1)^{\left(\frac{s_i^l}{2}\right)}\right)^{\left(1/(T+1)\right)} = 1$$

Test 4 – Continuity, positivity and normalization test

Definition - continuity, positivity and normalization test postulates that:

$P^{0,t}$ is a positive and a continuous function of prices and quantities vectors and $P^{0,0} = 1$

Proof - ITGEKS method passes continuity, positivity and normalization test as follows:

If a sample S is defined, it can be claimed that WTPD price imputations for missing products of a sample S are positive. This is since $\hat{p}_i^t = exp\left(\hat{\alpha} + \hat{\delta}^t + \hat{\gamma}_i\right)$ and $\hat{p}_i^0 = exp(\hat{\alpha} + \hat{\gamma}_i)$ are always positive. Moreover, if a sample AS is also defined, which is obtained after WTPD price imputations on a sample S, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$. Crucially, since GEKS method passes continuity, positivity and normalization test, it can be claimed that $P_{ITGEKS}^{0,t}(S)$ is a positive and a continuous function of prices and quantities.

$P_{ITGEKS}^{0,0} = 1$ holds since $P_{IT}^{0,t}$passes time reversal test.

Test 5 – Proportional prices test

Definition - proportional prices test postulates that:

if there exists λ^t such that $p_i^t = \lambda^t p_i^0$ for all $i = 1, \dots, n$,
then $P^{0,t} = \lambda^t$ for all $t = 0, \dots, T$

Proof - ITGEKS method passes proportional prices test as follows:

If a sample $S = \{p^0, \dots, \alpha^k p^0, \dots, \alpha^T p^0, q^0, \dots, q^k, \dots, q^T\}$ is defined, it can be claimed that WTPD price imputations for missing products of a sample S are different only by λ^t in each month. This is since WTPD method passes proportional prices test such that $P_{WTPD}^{0,t}(S) = \lambda^t$ (time dummy components of price imputations are different by λ^t in each month) and since no other WTPD price imputations components or regression weights are different. Moreover, if a sample AS is also defined, which is obtained after WTPD price imputations on a sample S, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$. Crucially, since GEKS method passes proportional prices test such that $P_{GEKS}^{0,t}(AS) = \lambda^t$, it can be claimed that ITGEKS method passes proportional prices test such that $P_{ITGEKS}^{0,t}(S) = \lambda^t$.

Test 6 – Homogeneity in quantities test

Definition – homogeneity in quantities test postulates that:

if, for any month k, there exists μ such that $\tilde{q}_i^k = \mu q_i^k$ for all $i = 1, \ldots, n$, then
$$P^{0,t}(p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T) = P^{0,t}\left(p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, \tilde{q}^k, \ldots, q^T\right)$$
for all $t = 0, \ldots, T$

Proof - ITGEKS method passes homogeneity in quantities test as follows:

If 2 samples $S = \{p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T\}$ and $S^* = \{p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, \tilde{q}^k, \ldots, q^T\}$, which are different only by μ such that $\tilde{q}_i^k = \mu q_i^k$, are defined, it can be claimed that WTPD price imputations for missing products of samples S and S^* are the same. This is since WTPD method passes homogeneity in quantities test such that $P_{WTPD}^{0,t}(S) = P_{WTPD}^{0,t}(S^*)$ (time dummy components of price imputations are the same for both samples) and since no other WTPD price imputations components or regressions weights are different. Moreover, if samples AS and AS^* are also defined, which are obtained after WTPD price imputations on samples S and S^*, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$ and $P_{ITGEKS}^{0,t}(S^*) = P_{GEKS}^{0,t}(AS^*)$. Crucially, since GEKS method passes homogeneity in quantities test such that $P_{GEKS}^{0,t}(AS) = P_{GEKS}^{0,t}(AS^*)$, it can be claimed that ITGEKS method passes homogeneity in quantities test such that $P_{ITGEKS}^{0,t}(S) = P_{ITGEKS}^{0,t}(S^*)$.

Test 7 – Homogeneity in prices test

Definition – homogeneity in prices test postulates that:

if, for any month $k \neq 0$, there exists μ such that $\tilde{p}_i^k = \mu p_i^k$ for all $i = 1, \ldots, n$, then
$$\mu P^{0,k}(p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T) = P^{0,k}\left(p^0, \ldots, \tilde{p}^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T\right)$$

Proof - ITGEKS method passes homogeneity in prices test as follows:

If 2 samples $S = \{p^0, \ldots, p^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T\}$ and $S^* = \{p^0, \ldots, \tilde{p}^k, \ldots, p^T, q^0, \ldots, q^k, \ldots, q^T\}$, which are different only by μ such that $\tilde{p}_i^k = \mu p_i^k$, are defined, it can be claimed that WTPD price imputations for missing products of samples S and S^* are the same with an exception of price imputations in a month k. These price imputations are inflated by μ in a sample S^*. This is since WTPD method passes homogeneity in prices test such that $\mu P_{WTPD}^{0,k}(S) = P_{WTPD}^{0,k}(S^*)$ (time dummy components of price imputations are increased by ln(μ) in a month k in a sample S^*) and since no other WTPD price imputations components or regressions weights are different. Moreover, if samples AS and AS^* are also defined, which are obtained after WTPD price imputations on samples S and S^*, it can be claimed that $P_{ITGEKS}^{0,k}(S) = P_{GEKS}^{0,k}(AS)$ and $P_{ITGEKS}^{0,k}(S^*) = P_{GEKS}^{0,k}(AS^*)$. Crucially, since GEKS method passes homogeneity in prices test such that $\mu P_{GEKS}^{0,k}(AS) = P_{GEKS}^{0,k}(AS^*)$, it can be claimed that ITGEKS method passes homogeneity in prices test such that $\mu P_{ITGEKS}^{0,k}(S) = P_{ITGEKS}^{0,k}(S^*)$.

Test 8 – Commensurability test

Definition – commensurability test postulates that:

if there exists $\mu_i > 0$ such that $\tilde{p}_i^t = \mu_i p_i^t$ and $\tilde{q}_i^t = q_i^t/\mu_i$ for all $i = 1, \dots, n$, then $P^{0,t}(p^0, \dots, p^T, q^0, \dots, q^T) = P^{0,t}(\tilde{p}^0, \dots, \tilde{p}^T, \tilde{q}^0, \dots, \tilde{q}^T)$ for all $t = 0, \dots, T$

Proof - ITGEKS method passes commensurability test as follows:

If 2 samples $S = \{p^0, \dots, p^T, q^0, \dots, q^T\}$ and $S^* = \{\tilde{p}^0, \dots, \tilde{p}^T, \tilde{q}^0, \dots, \tilde{q}^T\}$, which are different only by μ_i such that $\tilde{p}_i^t = \mu_i p_i^t$ and $\tilde{q}_i^t = q_i^t/\mu_i$ for all $i = 1, \dots, n$ and for all $t = 0, \dots, T$, are defined, it can be claimed that WTPD price imputations for missing products of samples S and S^* are different. These price imputations are inflated by μ_i in a sample S^*. This is since WTPD method passes commensurability test such that $P_{WTPD}^{0,t}(S) = P_{WTPD}^{0,t}(S^*)$ (time dummy components of price imputations are the same for both samples) and since only product dummy components and no other WTPD price imputations components or regressions weights are different. These product dummy components are increased by $\ln(\mu_i)$ in a sample S^*. Moreover, if samples AS and AS^* are also defined, which are obtained after WTPD price imputations on samples S and S^*, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$ and $P_{ITGEKS}^{0,t}(S^*) = P_{GEKS}^{0,t}(AS^*)$. Crucially, since GEKS method passes commensurability test such that $P_{GEKS}^{0,t}(AS) = P_{GEKS}^{0,t}(AS^*)$, it can be claimed that ITGEKS method passes commensurability test such that $P_{ITGEKS}^{0,t}(S) = P_{ITGEKS}^{0,t}(S^*)$.

Test 9 – Symmetry in the treatment of time periods test

Definition – symmetry in the treatment of time periods test postulates that:

reordering of the time periods doesn't change the price index between them

Proof - ITGEKS method passes symmetry in the treatment of time periods test as follows:

If 2 samples of seasonal products $S = \{p^0, \dots, p^k, p^l, \dots, p^T, q^0, \dots, q^k, q^l \dots, q^T\}$ and $S^* = \{p^0, \dots, p^l, p^k, \dots, p^T, q^0, \dots, q^l, q^k \dots, q^T\}$, which are different only by month ordering, are defined, it can be claimed that WTPD price imputations for missing products of samples S and S^* are the same. This is since WTPD method passes symmetry in the treatment of time periods test such that $P_{WTPD}^{0,t}(S) = P_{WTPD}^{0,t}(S^*)$ (time dummy components of price imputations are the same for both samples) and since no other WTPD price imputations components or regressions weights are different. Moreover, if samples AS and AS^* are also defined, which are obtained after WTPD price imputations on samples S and S^*, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$ and $P_{ITGEKS}^{0,t}(S^*) = P_{GEKS}^{0,t}(AS^*)$. Crucially, since GEKS method passes symmetry in the treatment of time periods test such that $P_{GEKS}^{0,t}(AS) = P_{GEKS}^{0,t}(AS^*)$, it can be claimed that ITGEKS method passes symmetry in the treatment of time periods test such that $P_{ITGEKS}^{0,t}(S) = P_{ITGEKS}^{0,t}(S^*)$.

Test 10 – Symmetry in the treatment of time products test

Definition – symmetry in the treatment of time products test postulates that:

reordering of the products doesn't change the price index result

Proof - ITGEKS method passes symmetry in the treatment of time products test as follows:

If 2 samples of seasonal products $S = \{p^0, \dots, p^T, q^0, \dots, q^T\}$ and $S^* = \{p^0, \dots, \hat{p}^T, q^0, \dots, \hat{q}^T\}$, which are different only by time product ordering in month T, are defined, it can be claimed that WTPD price imputations for missing products of samples S and S^* are the same. This is since WTPD method passes symmetry in the treatment of time products test such that $P_{WTPD}^{0,t}(S) = P_{WTPD}^{0,t}(S^*)$ (time dummy components of price imputations are the same for both samples) and since no other WTPD price imputations components or regressions weights are different. Moreover, if samples AS and AS^* are also defined, which are obtained after WTPD price imputations on samples S and S^*, it can be claimed that $P_{ITGEKS}^{0,t}(S) = P_{GEKS}^{0,t}(AS)$ and $P_{ITGEKS}^{0,t}(S^*) = P_{GEKS}^{0,t}(AS^*)$. Crucially, since GEKS method passes symmetry in the treatment of time products test such that $P_{GEKS}^{0,t}(AS) = P_{GEKS}^{0,t}(AS^*)$, it can be claimed that ITGEKS method passes symmetry in the treatment of time products test such that $P_{ITGEKS}^{0,t}(S) = P_{ITGEKS}^{0,t}(S^*)$.