ON THE USE OF UNIT VALUE INDICES AS CONSUMER PRICE SUBINDICES

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Abstract: This paper reviews the properties of the unit value index and discusses the question when it is warranted to use unit value indices as subindices of a Consumer Price Index.

1. Introduction and summary

Recently the interest into the calculation methods that are appropriate at the lowest level of aggregation of a Consumer Price Index (CPI) has increased. The International Conference on Price Indices, 31 October - 2 November 1994, organized by Statistics Canada, was largely devoted to this topic.

Diewert (1995) argues that at the lowest level of aggregation—that is the step before 'the first step' of Balk (1994)—the unit value and the total quantity sold constitute the appropriate price and quantity information. But what precisely is this lowest level? Diewert (1995) typically thinks of a 'homogeneous' commodity in a specific outlet, but it could also be a 'homogeneous' commodity in all outlets within a certain market area. He adds that the time period over which unit value and total quantity is calculated is important, by referring to the inflationary environment and the phenomenon of 'time of the day/week' commodities.

An important question is thus: when is a commodity (group)—that is, a set of economic transactions—sufficiently 'homogeneous' to warrant the use of unit values? Do we need to restrict to 'homogeneous' commodities in order to use unit values? In this paper we will pursue whether theory provides guidance to these problems.

[1] The views expressed in this paper are those of the author and do not necessarily reflect the policies of Statistics Netherlands.

In the next section we present a formal definition of the unit value index and review its properties. It appears that the unit value index has two defects, namely that it does not satisfy the proportionality test and that it is sensitive to the units of measurement. Section 3 presents an expression for the unit value bias that can be used in a strategy directed at finding 'homogeneity'

Then section 4 considers the appropriateness of unit values from the micro-economic perspective of the cost-of-living index theory. It appears that using the unit value index of a commodity group as a cost-of-living subindex is equivalent to maintaining that the corresponding group utility function is the simple sum function. In such a case, however, the unit value index is equal to each single price ratio. In practice there may be small distortions from such an homogeneity. Section 5 then demonstrates that the unit value index attains a higher precision than the observation of a single price ratio.

2. **Axiomatic considerations**

We consider a commodity group A consisting of commodities 1, . . ., N. The base period vectors of prices and quantities are $p^0 \equiv (p_1^0,\ldots,p_N^0)$ and $x^0 \equiv (x_1^0,\ldots,x_N^0)$ respectively. For the comparison period they are $p^1 \equiv (p_1^1,\ldots,p_N^1)$ and $x^1 \equiv (x_1^1,\ldots,x_N^1)$ respectively. Prices and quantities are assumed to be strictly positive. The expenditure on A in period t is $\sum_{n=1}^{N} p_n^t x_n^t \equiv p^t x^t \ (t=0,1)$ · The unit value of A in period t is the expenditure divided by the total quantity, thus $\sum_{n=1}^{N} p_n^t x_n^t / \sum_{n=1}^{N} x_n^t = p^t x^t / \iota x^t$, where ι is the N-dimensional vector of ones. It is presupposed that it somehow makes sense to add the quantities of A-commodities.

The unit value index, introduced by Drobisch (1871), for period 1 relative to period O is defined as

(1) $$U(p^1, x^1, p^0, x^0) \equiv (p^1 x^1 / \iota x^1)/(p^0 x^0 / \iota x^0)\ .$$

In this section we will see to which extent the unit value index satisfies the axioms (tests) for a price index (see Eichhorn and Voeller 1976, Balk 1995).

Al. Monotonicity. The unit value index is monotonous in prices, that is

$U(p^2, x^1, p^0, x^0) > U(p^1, x^1, p^0, x^0)$ if[2] $p^2 \geq p^1$ and

$U(p^1, x^1, p^2, x^0) < U(p^1, x^1, p^0, x^0)$ if $p^2 \leq p^0$.

A2. Linear homogeneity. The unit value index is linearly homogeneous in comparison period prices, that is $U(\lambda p^1, x^1, p^0, x^0) = \lambda U(p^1, x^1, p^0, x^0)$ $(\lambda > 0)$

A3. Identity. In general $U(p^0, x^1, p^0, x^0) \neq 1$, that is, if comparison prices equal base period prices the unit value index may show up with a value different from 1. This is the reason why Laspeyres (1871) rejected the unit value index. It is easy to show that

(2) $\quad U(p^0, x^1, p^0, x^0) = 1$ if and only if $x^1 / \iota x^1 = x^0 / \iota x^0$,

that is, if the relative quantities do not change.

A4. Homogeneity of degree zero. The unit value index is homogeneous of degree zero in prices and quantities, that is $U(\lambda p^1, x^1, \lambda p^0, x^0) = U(p^1, x^1, p^0, x^0)$ and

$U(p^1, \lambda x^1, p^0, \lambda x^0) = U(p^1, x^1, p^0, x^0)$ $(\lambda > 0)$

A5. Dimensional invariance. Let Λ be a diagonal matrix with strictly positive elements. Then it is easy to verify that unless all elements of Λ are equal

$U(\Lambda p^1, \Lambda^{-1} x^1, \Lambda p^0, \Lambda^{-1} x^0) \neq U(p^1, x^1, p^0, x^0)$. Thus the unit value index is sensitive to the choice of the units of measurement. As Parniczky (1974) observes, this choice may have a substantial impact on the outcome.

A6. Proportionality. A proportional change of all prices does in general not result in the same proportional change of the unit value index. In fact, (2) can be generalized to

(3) $\quad U(\lambda p^0, x^1, p^0, x^0) = \lambda$ if and only if $x^1 / \iota x^1 = x^0 / \iota x^0$.

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[2] $p^2 \geq p^1$ means that $p_n^2 \geq p_n^1$, for n=1,...,N but $p^2 \neq p^1$.

T1. Transitivity. The unit value index is transitive, that is $U(p^2, x^2, p^1, x^1)U(p^{1,}, x^1, p^0, x^0) = U(p^2, x^2, p^0, x^0)$. One also says that the unit value index satisfies the circular test.

T2. Time reversal. The unit value index satisfies the time reversal test, that is $U(p^0, x^0, p^1, x^1) = 1/U(p^1, x^1, p^0, x^0)$.

T3. Product test. There exists a quantity index $Q(p^1, x^1, p^0, x^0)$ such that $U(p^1, x^1, p^0, x^0)Q(p^{1,}, x^1, p^0, x^0) = p^1x^1 / p^0x^0$, the value ratio. The implied quantity index is

(4) $$Q_{UV}(p^1, x^1, p^0, x^0) = \iota x^1 / \iota x^0 \ .$$

This quantity index is monotonous in quantities, linearly homogeneous in comparison period quantities, satisfies the identity axiom, is homogeneous of degree zero in quantities, but not dimensionally invariant. It satisfies the proportionality axiom, the circular test, and the time reversal test.

This completes our overview of the properties of the unit value index. In view of the fact that it does not satisfy the dimensionality axiom *A5*, and the proportionality axiom *A6*, *the unit value index can not be called a price index*. In this respect the unit value index differs from the Laspeyres price index,

(5) $$P_L(p^1, x^1, p^0, x^0) = p^1x^0 / p^0x^0 \ ,$$

the Paasche price index,

(6) $$P_P(p^1, x^1, p^0, x^0) = p^1x^1 / p^0x^1 \ ,$$

and the Fisher price index,

(7) $$P_F(p^1, x^1, p^0, x^0) = [P_L(p^1, x^1, p^0, x^0) P_P(p^1, x^1, p^0, x^0)]^{1/2} \ .$$

This is however a purely theoretical result. In practice there may occur situations in which the unit value index behaves like a genuine price index. This is the topic of the next section.

3. Empirical considerations

Consider the ratio of the unit value index to a price index. The difference between this ratio and 1 is called the unit value bias, Parniczky (1974) considered the unit value bias with respect to the Paasche price index. Here we develop an expression for the unit value bias with respect to the Fisher price index. The latter index is chosen because it is a superlative index .

Straightforward algebraic manipulations yield the following expression,

(8)
$$U(p^1, x^1, p^0, x^0) / P_F(p^{1,}, x^1, p^0, x^0) = [(1 + rel\,\mathrm{cov}_0(p_n^1, x_n^1 / x_n^0))(1 + rel\,\mathrm{cov}(p_n^0, x_n^1 / x_n^0))]^{1/2}$$

where the relative covariances are defined by

(9)

$$rel\,\mathrm{cov}_0(p_n^t, x_n^1 / x_n^0) \equiv \sum_n x_n^0 (p_n^t - p^t x^0 / \iota x^0)(x_n^1 / x_n^0 - \iota x^1 / \iota x^0) / (p^t x^0 (\iota x^1 / \iota x^o)) \ (t = 0,1) \ .$$

We can now draw the following conclusion. The unit value bias will be equal to zero if one or more of the following situations occur:

- all base period prices p_n^0 are equal to each other and all comparison period prices p_n^1 are equal to each other;
- all quantity relatives x_n^1 / x_n^0 are equal to each other;
- there is, both in base period and in comparison period, no correlation between prices and quantity relatives.

If there is a non-zero unit value bias we could opt for the following strategy. Divide the commodity group A into subgroups, calculate for each subgroup the unit value index, and aggregate these unit value indices with help of, say, the Fisher price index. Parniczky (1974) shows that such a strategy not necessarily diminishes the unit value bias. The unit value bias can be split into two parts, a within-subgroups contribution and a between-subgroups contribution. The latter contribution disappears due to our strategy, but if both contributions are of opposite sign there can result a larger bias (in absolute value)

Thus any desaggregation strategy must depend on knowledge about the relative importance of the bias components.

4. Micro-economic considerations

We now turn to some micro-economic considerations. Suppose that the consumer's utility function is strictly separable in a number of commodity groups, and let A be one of those groups. Then there exists a partial cost function *C(p,u)* which gives the minimum expenditure on A-commodities for obtaining group utility level $u = F(x)$ when the prices of the A-commodities are *p*. The partial cost-of-living index for group A is then defined as

(10) $$P(p^1, p^0; u) = C(p^1, u) / C(p^0, u)$$

(see Balk 1990) . In this section we consider under which circumstances the partial cost-of-living index is equal to the unit value index (1)

We assume that

(11) $$p^t x^t = C(p^t, F(x^t)) \;\; (t = 0,1) \;,$$

that is in both periods the actual expenditure on A-commodities is the minimum expenditure for obtaining the implied group utility level. Let

(12) $$P(p^1, p^0; u) = U(p^1, x^1, p^0, x^0) \text{ for some value of } u.$$

If we substitute (1), (10) and (11) into (12) and rearrange, we obtain

(13) $\iota x^0 / \iota x^1 = [C(p^1, u) / C(p^0, u)][C(p^0, F(x^0)) / C(p^1, F(x^1))]$.

This must hold for all p^1, x^1, p^0, x^0 . Thus in particular for $p^0 = p^1 = p$

(14) $\iota x^0 / \iota x^1 = C(p, F(x^0)) / C(p, F(x^1))$,

or

(15) $C(p, F(x^0)) / \iota x^0 = C(p, F(x^1)) / \iota x^1$.

Since (15) must hold for all x^1, x^0, we conclude that for all p, x

(16) $C(p, F(x)) / \iota x = C(p, F(\iota)) / \iota\iota = C(p, F(\iota)) / N \equiv c(p)$

Thus *c(p)* is the 'average price' of the *N* commodities. For fixed *p* *C(p,F(x))* is a money-metric utility function. This function has apparently the following form

(17) $C(p, F(x)) = \iota x c(p)$.

But this implies that the underlying preference ordering can also be represented by the utility function

(18) $F(x) = \iota x$,

that is the utility of $x_1, ..., x_N$ is given by the simple sum $\sum_{n=1}^{N} x_n$.

Reversely, let (18) hold. By simple geometry, see Figure 1, it is clear that

(19) $C(p,u) = \min_x\{px \mid \iota x \geq u\} = u \min_n\{p_n\}$,

and thus

(20) $P(p^1, p^0; u) = \min_n\{p_n^1\} / \min_n\{p_n^0\}$.

Without loss of generality we can suppose that in each period all A-commodities are consumed in strictly positive quantities. Then it must be the case that in period 0

(21) $p_1^0 = ... = p_n^0$ and $u^0 = p^0 x^0 / p_N^0 \quad (n = 1,..., N)$,

and in period 1

(22) $p_1^1 = ... = p_N^1$ and $u^1 = p^1 x^1 / p_N^1 \quad (n = 1,..., N)$.

Thus

(23)

$$U(p^1, x^1, p^0, x^0) = (p^1 x^1 / p^0 x^0)(\iota x^0 / \iota x^1) = (p_n^1 u^1 / p_n^0 u^0)(u^0 / u^1) = p_n^1 / p_n^0 = P(p^1, p^0; u).$$

The foregoing can be summarized as follows.

THEOREM: IF IN BASE AND COMPARISON PERIOD THE EXPENDITURES ON THE COMMODITY GROUP ARE OPTIMAL WITH RESPECT TO THE PREVAILING PRICES THEN THE PARTIAL COST-OF-LIVING INDEX IS EQUAL TO THE UNIT VALUE INDEX IF AND ONLY IF THE UNDERLYING PREFERENCE ORDERING CAN BE REPRESENTED BY THE SIMPLE SUM UTILITY FUNCTION.

Notice that the simple sum utility function has the property that the marginal utility of each commodity is equal to 1, that is

(24) $\partial F(x)/\partial x_n = 1 \quad (n = 1,\ldots, N)$.

5. Sampling considerations

The proof of the foregoing theorem makes clear that if the unit value index is appropriate for a certain commodity group then it is equal to each single price ratio, and all those price ratios are equal. Thus the observation of only one commodity suffices to calculate the price index. In practice, however, there may occur small distortions. Let these be modelled as follows:

(25) $p_n^t = p^t(1+\varepsilon_n^t)$

where $E\varepsilon_n^t = 0$, $E(\varepsilon_n^t)^2 = \sigma^2$, $E\varepsilon_n^0\varepsilon_n^1 = 0$, $E\varepsilon_n^t = 0$, and

$E\varepsilon_n^t\varepsilon_n^t = 0 \quad (n, n' = 1,\ldots, N;\ n \neq n'; t = 0,1)$

Then

(26) $p_n^1 / p_n^0 = (p^1/p^0)(1+\varepsilon_n^1)/(1+\varepsilon_n^0) = (p^1/p^0)(1+\varepsilon_n^1 - \varepsilon_n^0)$,

which implies that the mean square error of a single price ratio is equal to

(27) $E(p_n^1/p_n^0 - p^1/p^0)^2 \cong E(\varepsilon_n^1 - \varepsilon_n^0)^2 = 2\sigma^2$

For the unit value index we obtain similarly

(28)
$$\begin{aligned} &U(p^1, x^1, p^0, x^0) = \\ &(p^1/p^0)(1+\textstyle\sum_n \varepsilon_n^1 x_n^1 / \sum_n x_n^1)/(1+\sum_n \varepsilon_n^0 x_n^0 / \sum_n x_n^0) \cong \\ &(p^1/p^0)(1+\textstyle\sum_n \varepsilon_n^1 x_n^1 / \sum_n x_n^1 - \sum_n \varepsilon_n^0 x_n^0 / \sum_n x_n^0) \cong \\ &(p^1/p^0)(1+(1/N)\textstyle\sum_n \varepsilon_n^1 - (1/N)\sum_n \varepsilon_n^0) \end{aligned}$$

where the final approximation is based on the assumption that the share of each commodity in the total quantity of the group is of the order 1/N.

Based upon (28) we obtain for the mean square error of the unit value index

$$(29)\quad E(U(p^1, x^1, p^0, x^0) - p^1/p^0)^2 \cong (2/N)\sigma^2$$

Thus the unit value index attains a higher precision than a single price ratio. This corresponds to Diewert's (1995) remark that "It should be evident that a unit value for the commodity provides a more accurate summary of an average transaction price than an isolated price quotation."

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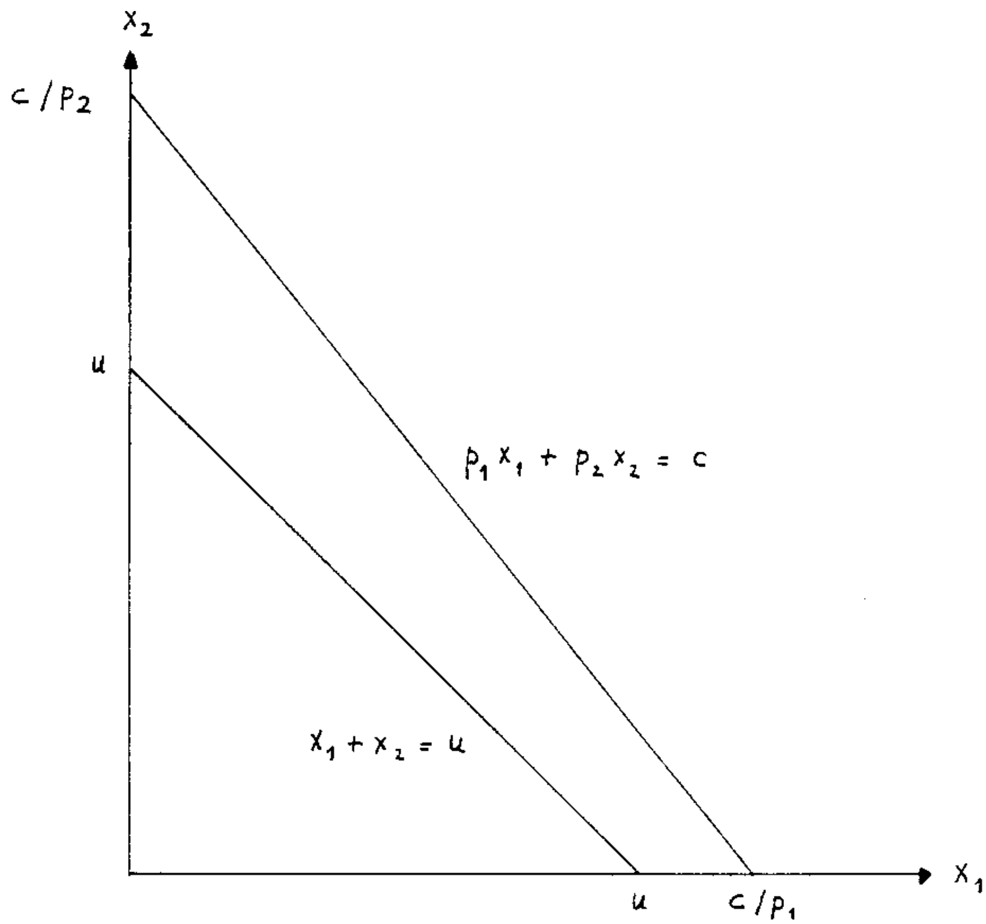

Figure 1. The cost minimization problem (19)