The general class of multilateral indices and its two special cases

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Agenda

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Basic notations and considered multilateral indices

Multilateral index methods originate in comparisons of price levels across countries or regions. Commonly known methods include the **GEKS** method (Gini, 1931; Eltetö and Köves, 1964), the **Geary-Khamis** method (Geary, 1958; Khamis, 1972) - see also (Chessa, 2016; Chessa et al., 2017), the **CCDI** method (Caves et al., 1982) or the **Time Product Dummy Methods** (de Haan and Krsinich, 2018). Before we present the proposed multilateral price indices let us denote sets of homogeneous products belonging to the same product group in months 0 and t by G_0 and G_t respectively, and let $G_{0,t}$ denote a set of matched products in both moments 0 and t. Although, in general, the item universe may be very dynamic in the scanner data case, we assume that there exits at least one product being available during the whole time interval $[0, T]$. Let p_i^τ and q_i^τ denote the price and quantity of the i-th product at time τ and $N_{0,t} = \text{card}\, G_{0,t}$.

The SPQ method

The idea of the SPQ multilateral price index is based on the relative price and quantity dissimilarity measure Δ_{SPQ} (Diewert, 2020). The price dissimilarity measure is used to link together the bilateral Fisher indices according to the following algorithm:
(1) set $P_{SPQ}^{0,0} = 1$ for the period 0 (base month) and calculate $P_{SPQ}^{0,1} = P_F(p^0, p^1, q^0, q^1)$ for the period 1;
(2) For the period 2 define $P_{SPQ}^{0,2} = P_F(p^0, p^2, q^0, q^2)P_{SPQ}^{0,0}$ if $min\{\Delta_{SPQ}(p^0, p^2, q^0, q^2), \Delta_{SPQ}(p^1, p^2, q^1, q^2)\}$ is equal $\Delta_{SPQ}(p^0, p^2, q^0, q^2)$ or take $P_{SPQ}^{0,2} = P_F(p^1, p^2, q^1, q^2)P_{SPQ}^{0,1}$ otherwise;
(3) Continue this procedure in the same manner, i.e. for the period t, let r^* be such a period that $min_r\{\Delta_{SPQ}(p^r, p^t, q^r, q^t), r = 1, 2, ..., t-1\} = \Delta_{SPQ}(p^{r^*}, p^t, q^{r^*}, q^t)$. For the considered period define $P_{SPQ}^{0,t} = P_F(p^{r^*}, p^t, q^{r^*}, q^t)P_{SPQ}^{0,r^*}$.

Criteria in the multilateral index method selecting

- Axiomatic approach
- Economic approach
- Stochastic approach
- New variant of the stochastic approach (Białek and Bobel, 2019; Białek and Beręsewicz, 2021)
- Time-consuming approach (Białek and Beręsewicz, 2021)

Axioms for multilateral index methods

Let P and Q denote all prices and quantities observed in the time interval $[0, T]$, i.e. $P = [p^0, p^1, ..., p^T]$, $Q = [q^0, q^1, ..., q^T]$, where p^t and q^t mean the vector of prices and the vector of quantities of products sold at time t, respectively. Let us denote by $P^{0,t}(P, Q)$ the considered multilateral price index defined for the entire time window $[0, T]$. The list of potential tests for that index is as follows:

Transitivity

The transitivity means that $P^{0,t}(P, Q) = P^{0,s}(P, Q)P^{s,t}(P, Q)$ for any $0 \leq s < t \leq T$.

Identity

This property means that the index equals identity if all prices revert back to their initial level, i.e. if it holds that $p_i^t = p_i^0$ for $i \in G_{0,t}$ then $P^{0,t}(P, Q) = 1$. We assume here that the item universe is the same at periods 0 and t.

Multi period identity test

This property means that if all prices and quantities revert back to their initial level, the chained index will equal the unity, i.e. if it holds that $p_i^t = p_i^0$ and $q_i^t = q_i^0$ for $i \in G_{0,t}$ then we obtain $P^{0,1}(P, Q) \times P^{1,2}(P, Q) \times .... \times P^{t-1,t}(P, Q) = 1$. We assume here that the item universe is the same at periods 0 and t.

Fixed basket test

If $G_0 = G_t$ and $q_i^0 = q_i^t = q_i$ for $i \in G_{0,t}$, then $P^{0,t}(P, Q) = \frac{\sum_{i \in G_{0,t}} p_i^t q_i}{\sum_{i \in G_{0,t}} p_i^0 q_i}$.

Upper bound test

If $G_0 \subset G_t$ and $p_i^t \leq p_i^0$ for all $i \in G_0$, then $P^{0,t}(P, Q) \leq 1$.

Lower bound test

If $G_t \subset G_0$ and $p_i^t \geq p_i^0$ for all $i \in G_t$, then $P^{0,t}(P, Q) \geq 1$.

Responsiveness test

For $G_0 \neq G_t$, if $p_i^t = p_i^0$ for all $i \in G_{0,t}$, then $P^{0,t}(P, Q)$ cannot always equal one, regardless of sets: $G_0 \setminus G_t$ and $G_t \setminus G_0$.

Continuity, positivity and normalization

$P^{0,t}(P, Q)$ is a positive and continuous function of prices and quantities, $P^{0,0}(P, Q) = 1$.

Price proportionality

If all prices are proportional in compared periods 0 and t, i.e. $p_i^t = kp_i^0$ for all $i \in G_{0,t}$ and some positive k, then the price index depends only on this proportion: $P^{0,t}(P, Q) = k$. We assume here that the item universe is the same at periods 0 and t.

Homogeneity in quantities

Rescaling the quantities in any s-th period does not influence on the price index, i.e. for any positive k it holds that $P^{0,t}(P, q^0, ..., kq^s, ..., q^t) = P^{0,t}(P, q^0, ..., q^s, ..., q^t)$.

Homogeneity in prices

Rescaling the prices in the current period changes the price index by the same proportion, i.e. for any positive k it holds that $P^{0,t}(p^0, p^1, ..., kp^t, Q) = kP^{0,t}(p^0, p^1, ..., p^t, Q)$.

Commensurability

Changing the units in which prices and quantities are expressed does not change the price index. In other words, if for each time moment $s \in [0, T]$ we have $\tilde{p}_i^s = \lambda_i p_i^s$ and $\tilde{q}_i^s = \frac{q_i^s}{\lambda_i}$ for all $i \in G_s$ ($\lambda_i > 0$), then $P^{0,t}(\tilde{P}, \tilde{Q}) = P^{0,t}(P, Q)$.

Proposition of the general class of semi-GEKS indices

Let us consider the following (general) class of GEKS-type indices:

$$P_{GS-GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P^{\tau,t}}{P^{\tau,0}} \right)^{\frac{1}{T+1}}. \quad (1)$$

where $P^{\tau,s}$ is the choosen bilateral price index formula (for $s = 0, t$). Our proposal assumes that the formula $P^{\tau,s}$, which compares the current period s with the base period τ, can be written in the following form:

$$P^{\tau,s} = f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s}) \quad (2)$$

where a function $f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$ takes into account products from the $G_{\tau,s}$ set.

Proposition of the general class of semi-GEKS indices

We have a list of minimal requirements for the function $f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$: **R1**) It must be a positive and continuous function of its arguments and $f_{G_{\tau}}(q^{\tau}, p^{\tau}, p^{\tau}) = 1$; **R2**) The *proportionality* in current prices and *inverse proportionality* in base prices must hold (*homogeneity* of degree $+1$ in current prices and homogeneity of degree -1 in base prices), i.e. we expect that $f_{G_{\tau,s}}(q^{\tau}, mp^{\tau}, kp^{s}) = \frac{k}{m} f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$; **R3**) The only possible reaction to identical quantity changes is no reaction, i.e. $f_{G_{\tau,s}}(kq^{\tau}, p^{\tau}, p^{s}) = f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$; **R4**) For any diagonal matrix $\mathbf{L} = diag(\lambda_1, \lambda_2, ..., \lambda_{N_{\tau,s}})$, it must hold that $f_{G_{\tau,s}}(\mathbf{L}^{-1} q^{\tau}, \mathbf{L} p^{\tau}, \mathbf{L} p^{s}) = f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$; **R5**) For two different data sets $G^{*}_{\tau,s}$ and $G^{**}_{\tau,s}$ being subsets of $G_{\tau,s}$, where one is contained in the other (e.g. $G^{*}_{\tau,s} \subset G^{**}_{\tau,s}$), we obtain, in general, two different function values, i.e. $f_{G^{*}_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s}) \neq f_{G^{**}_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$. Please note that conditions **R1, R2, R4** make the formula $f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$ satisfy all axioms from the system of minimum requirements of price index by Martini (1992) (*identity, commensurability, linear homogeneity*).

Proposition of the general class of semi-GEKS indices

The requirements **R1**-**R5** are fundamental and they will be justified with respect to good axiomatic properties of the multilateral $GS - GEKS$ method in the next part of the paper (see Theorem 1). We will also consider the stronger version of the **R5** condition (not obligatory), which can be written as: **R6**) For two different data sets $G^{*}_{\tau,s}$ and $G^{**}_{\tau,s}$ such as $G^{*}_{\tau,s} \subset G^{**}_{\tau,s} \subset G_{\tau,s}$, we obtain $f_{G^{*}_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s}) \geq f_{G^{**}_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s})$. Finally, let us also take into consideration an additional requirement of *monotonicity*: **R7**) If $p^{s}_{i} \leq p^{s*}_{i}$ for any $i-$th product from $G_{\tau,s}$, then it holds that $f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s}) \leq f_{G_{\tau,s}}(q^{\tau}, p^{\tau}, p^{s*})$. As it will be shown (see Theorem 1), the requirements **R6** and **R7** are crucial with respect to the lower and upper bound tests.

Proposition of the general class of semi-GEKS indices

Taking bilateral index formula (2) as an input in the GEKS body we obtain the following form of the proposed general class of semi-GEKS indices:

$$P_{GS-GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{f_{G_{\tau,t}}(q^{\tau}, p^{\tau}, p^{t})}{f_{G_{\tau,0}}(q^{\tau}, p^{\tau}, p^{0})} \right)^{\frac{1}{T+1}}. \tag{3}$$

The following theorem can be proved:

Theorem

*Under restrictions R1-R5 each GS-GEKS index satisfies the transitivity, **identity**, multi period identity, responsiveness, continuity, positivity and normalization, commensurability, price proportionality, homogeneity in prices and homogeneity in quantitites tests. If the requirements R6 and R7 are additionally fulfilled, this index also satisfies the lower and upper bound tests.*

Proposition based on the Laspeyres formula (GEKS-L)

Let us define the function $f_{G_{\tau,s}}(q^\tau, p^\tau, p^s)$ as follows:

$$f^L_{G_{\tau,s}}(q^\tau, p^\tau, p^s) = \frac{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^\tau}, \tag{4}$$

where the "L" subscript refers to the Laspeyres (1871) formula. Putting (4) in formula GS-GEKS, we obtain

$$P^{0,t}_{GEKS-L} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^\tau}}{\frac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{5}$$

Please note that the GEKS-L index can be treated as the **generalization of the Fisher price index** formula ($P_F^{0,t}$) to the multi-period case. In fact, in a static item universe G observed over the two period time interval $[0, 1]$, we obtain

$$P_{GEKS-L}^{0,1} = \prod_{\tau=0}^{1} \left(\frac{\sum_{i \in G} q_i^{\tau} p_i^1}{\sum_{i \in G} q_i^{\tau} p_i^0}\right)^{\frac{1}{1+1}} = \left(\frac{\sum_{i \in G} q_i^0 p_i^1}{\sum_{i \in G} q_i^0 p_i^0} \times \frac{\sum_{i \in G} q_i^1 p_i^1}{\sum_{i \in G} q_i^1 p_i^0}\right)^{\frac{1}{2}} = P_F^{0,1}, \quad (6)$$

since $G_0 = G_1 = G_{0,1} = G$.

Proposition based on the geometric Laspeyres formula (GEKS-GL)

Let us define the function $f_{G_{\tau,s}}(q^\tau, p^\tau, p^s)$ as follows:

$$f^{GL}_{G_{\tau,s}}(q^\tau, p^\tau, p^s) = \prod_{i \in G_{\tau,s}} (\frac{p_i^s}{p_i^\tau})^{w_i^{\tau,s}(\tau)} \tag{7}$$

where

$$w_i^{\tau,s}(\tau) = \frac{q_i^\tau p_i^\tau}{\sum_{k \in G_{\tau,s}} q_k^\tau p_k^\tau}, \tag{8}$$

and the "GL" subscript refers to the geometric Laspeyres formula (von der Lippe, 2007). Putting (7) in the formula GS-GEKS, we obtain

$$P^{0,t}_{GEKS-GL} = \prod_{\tau=0}^{T} (\frac{\prod_{i \in G_{\tau,t}} (\frac{p_i^t}{p_i^\tau})^{w_i^{\tau,t}(\tau)}}{\prod_{i \in G_{\tau,0}} (\frac{p_i^0}{p_i^\tau})^{w_i^{\tau,0}(\tau)}})^{\frac{1}{T+1}}. \tag{9}$$

Please note that the GEKS-GL index can be treated as the generalization of the Törnqvist (1936) price index formula ($P_T^{0,t}$) to the multi-period case. In fact, in a static item universe G observed over the two period time interval $[0, 1]$, we obtain

$$P_{GEKS-GL}^{0,1} = \prod_{\tau=0}^{1} \left(\frac{\prod_{i \in G} (\frac{p_i^1}{p_i^\tau})^{w_i^{\tau,1}(\tau)}}{\prod_{i \in G} (\frac{p_i^0}{p_i^\tau})^{w_i^{\tau,0}(\tau)}} \right)^{\frac{1}{1+1}} = \prod_{\tau=0}^{1} \left(\prod_{i \in G} (\frac{p_i^1}{p_i^0})^{w_i(\tau)} \right)^{\frac{1}{2}}$$
$$= \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(0)}{2}} \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(1)}{2}} = \prod_{i \in G} (\frac{p_i^1}{p_i^0})^{\frac{w_i(0)+w_i(1)}{2}} = P_T^{0,1}, \tag{10}$$

since $G_0 = G_1 = G_{0,1} = G$, and consequently $w_i^{\tau,0}(\tau) = w_i^{\tau,1}(\tau) = w_i(\tau)$ for any τ.

The following theorem can be proved:

Theorem

GEKS-L and GEKS-GL indices satisfy the transitivity, ***identity****, multi period identity, responsiveness, continuity, positivity and normalization, commensurability, price proportionality, homogeneity in prices and homogeneity in quantitites tests.*

Other possible particular cases

It may be interesting to consider the following normalized versions of the GEKS-L and GEKS-GL indices:

$$P_{GEKS-NL}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{1}{N_{\tau,t}} \frac{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{t}}{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{\tau}}}{\frac{1}{N_{\tau,0}} \frac{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{0}}{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{\tau}}} \right)^{\frac{1}{T+1}}, \tag{11}$$

and

$$P_{GEKS-NGL}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{1}{N_{\tau,t}} \prod_{i \in G_{\tau,t}} \left(\frac{p_i^{t}}{p_i^{\tau}}\right)^{w_i^{\tau,t}(\tau)}}{\frac{1}{N_{\tau,0}} \prod_{i \in G_{\tau,0}} \left(\frac{p_i^{0}}{p_i^{\tau}}\right)^{w_i^{\tau,0}(\tau)}} \right)^{\frac{1}{T+1}}. \tag{12}$$

In analogy to the weighted GEKS index (Melser, 2018), it would be also possible to consider the following weighted versions of the GEKS-L and GEKS-GL indices:

$$P_{WGEKS-L}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\frac{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{t}}{\sum_{i \in G_{\tau,t}} q_i^{\tau} p_i^{\tau}}}{\frac{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{0}}{\sum_{i \in G_{\tau,0}} q_i^{\tau} p_i^{\tau}}} \right)^{v_\tau}, \tag{13}$$

and

$$P_{WGEKS-GL}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\prod_{i \in G_{\tau,t}} \left(\frac{p_i^t}{p_i^\tau}\right)^{w_i^{\tau,t}(\tau)}}{\prod_{i \in G_{\tau,0}} \left(\frac{p_i^0}{p_i^\tau}\right)^{w_i^{\tau,0}(\tau)}} \right)^{v_\tau}, \tag{14}$$

where the weights concerning the period τ could be defined as follows:

$$v_\tau = \frac{\sum_{i \in G_\tau} q_i^\tau p_i^\tau}{\sum_{\tau=0}^{T} \sum_{i \in G_\tau} q_i^\tau p_i^\tau}. \tag{15}$$

Empirical study

- Monthly scanner data:

Data from one retail chain in Poland on: **long grain rice** (subgroup of COICOP 5 group: 011111), **ground coffee** (subgroup of COICOP 5 group: 012111), **drinking yoghurt** (subgroup of COICOP 5 group: 011441) and **white sugar** (subgroup of COICOP 5 group: 011811) sold in over 210 outlets during the period from December 2019 to December 2020 (352705 records, which means 210 MB of data).

- Data processing:
  - Product classification in *PriceIndices*: **data_selecting()**, **data_classification()**;
  - Product matching in *PriceIndices*: **data_matching()** based on the Jaro-Winkler measure;
  - Data filtering in *PriceIndices*: **data_filtering()** with implemented *low sales filter*, *extreme price filter*, *dump price filter*;
- Index calculations in *PriceIndices*: **geksl()**, **geksgl()**, **geks()**, **gk()**, **tpd()**, **SPQ()**.

Empirical study - results

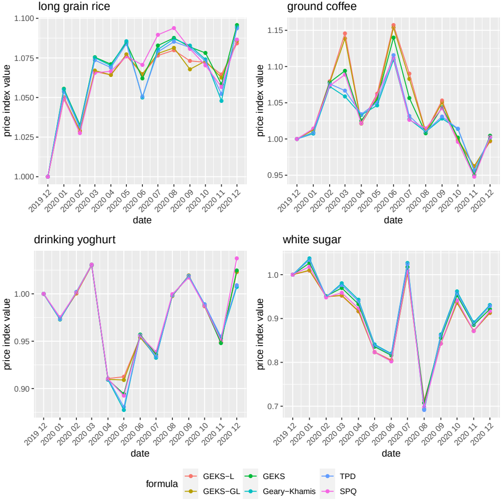

Figure: Comparison of selected multilateral indices for four homogeneous group of food products

Empirical study - results

Table: Mean absolute differences between considered price indices (*long grain rice*) [p.p]

distances	GEKS-L	GEKS-GL	GEKS	GK	TPD	SPQ
GEKS-L	0.000	0.124	0.644	0.751	0.609	0.445
GEKS-GL	0.124	0.000	0.605	0.713	0.571	0.457
GEKS	0.644	0.605	0.000	0.270	0.306	0.580
GK	0.751	0.713	0.270	0.000	0.143	0.707
TPD	0.609	0.571	0.306	0.143	0.000	0.617
SPQ	0.445	0.457	0.580	0.707	0.617	0.000

Table: Mean absolute differences between considered price indices (*ground coffee*) [p.p]

distances	GEKS-L	GEKS-GL	GEKS	GK	TPD	SPQ
GEKS-L	0.000	0.262	1.189	2.227	2.006	1.634
GEKS-GL	0.262	0.000	0.965	2.003	1.783	1.494
GEKS	1.189	0.965	0.000	1.203	0.984	0.773
GK	2.227	2.003	1.203	0.000	0.256	0.869
TPD	2.006	1.783	0.984	0.256	0.000	0.808
SPQ	1.634	1.494	0.773	0.869	0.808	0.000

Empirical study - results

Table: Mean absolute differences between considered price indices (*drinking yoghurt*) [p.p]

distances	GEKS-L	GEKS-GL	GEKS	GK	TPD	SPQ
GEKS-L	0.000	0.060	0.237	0.513	0.459	0.370
GEKS-GL	0.060	0.000	0.215	0.480	0.437	0.360
GEKS	0.237	0.215	0.000	0.361	0.337	0.207
GK	0.513	0.480	0.361	0.000	0.082	0.483
TPD	0.459	0.437	0.337	0.082	0.000	0.437
SPQ	0.370	0.360	0.207	0.483	0.437	0.000

Table: Mean absolute differences between considered price indices (*white sugar*) [p.p]

distances	GEKS-L	GEKS-GL	GEKS	GK	TPD	SPQ
GEKS-L	0.000	0.114	1.153	1.891	1.660	0.469
GEKS-GL	0.114	0.000	1.087	1.820	1.592	0.361
GEKS	1.153	1.087	0.000	0.757	0.523	0.914
GK	1.891	1.820	0.757	0.000	0.264	1.495
TPD	1.660	1.592	0.523	0.264	0.000	1.277
SPQ	0.469	0.361	0.914	1.495	1.277	0.000

Empirical study - results

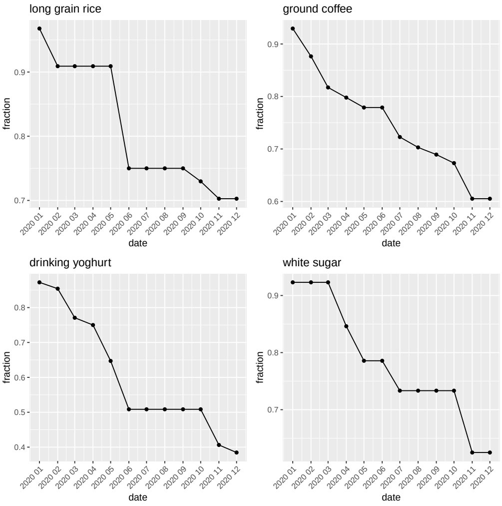

Figure: Monthly fractions of products remaining on sale since Dec 2019 calculated for the studied four product groups

Empirical study - results

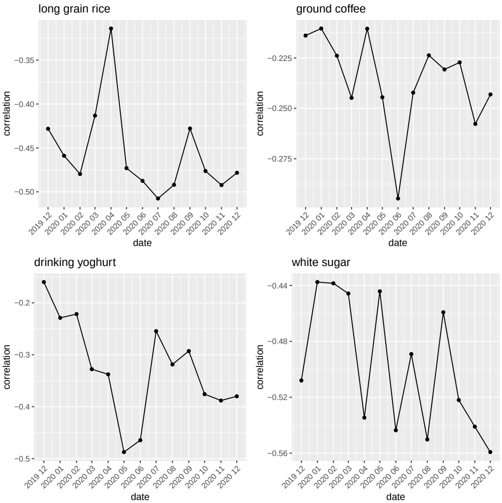

Figure: Monthly values of Pearson's correlation coefficient between prices and quantities calculated for the studied four product groups

Empirical study - results

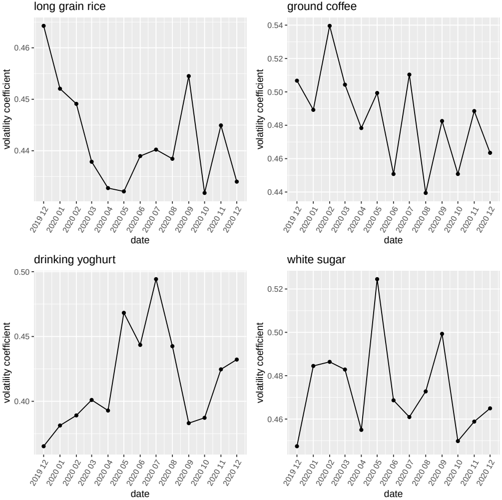

Figure: Monthly coefficients of variation of prices calculated for the studied four product groups

Empirical study - results

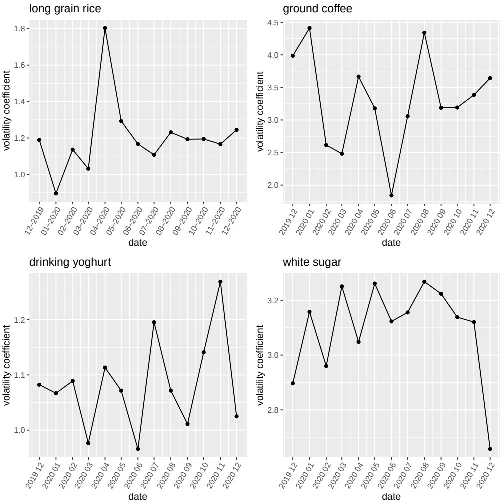

Figure: Monthly coefficients of variation of quantities calculated for the studied four product groups

Empirical study - results

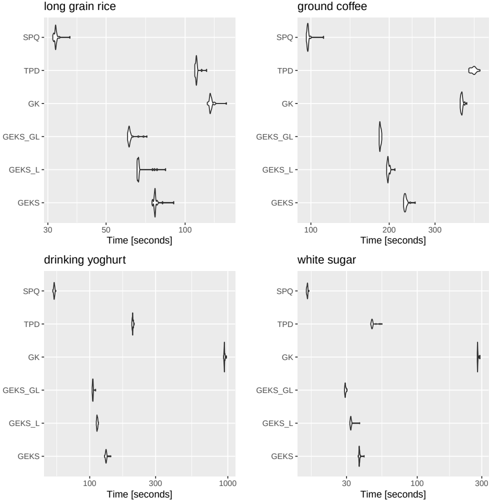

Figure: Monthly coefficients of variation of quantities calculated for the studied four product groups

Conclusions

The proposed **GEKS-L** and **GEKS-GL** indices seem to be promising because they:

- meet almost all required tests (axioms), including the *identity test*
- can be treated as generalizations of the Fisher and Törnqvist price index respectively
- need relatively short computing time.

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