Divisia Indexes and the Representative Consumer Problem

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April, 1998

“Empirical evidence is abundant that the Santa Claus hypothesis of homotheticity in tastes and in technical change is quite unrealistic. Therefore we must not be bemused by the undoubted elegances and richnesses of the homothetic theory. …we must accept the sad facts of life and be grateful for the more complicated procedures that economic theory devises.”

Samuelson and Swamy, 1974.

“It is evident that we shall get nowhere if all this individuality is to run riot. …. We get over the difficulty by shutting our eyes to it.”

In *Facts from Figures,* by M.J. Moroney, 1953.

Abstract— Divisia indexes play major roles in the measurement of productivity, price and output change. Paradoxically, however, the theoretical literature finds that the use of a Divisia index is justifiable only under an assumption that is often unrealistic. I replace this assumption with alternatives that plausibly describe most situations in which index numbers are needed. The Divisia quantity index then equals a weighted combination of the standard of living indexes implied by various possible reference price vectors. In special cases where the weights are everywhere non-negative, the Divisia price index equals a product of one cost of living index (CLI) based on the initial utility level and another based on the final utility level. The properties of Divisia indexes can be used to understand the properties of a CLI or chained index based on the aggregate demands of a group of consumers.

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* The views expressed in this paper are those of the author and should not be attributed to the FDIC or the Bureau of Labor Statistics, where I began this paper. I am grateful to Erwin Diewert, Zvi Griliches, Jack Reidhill and Kim Zieschang for helpful comments.

Divisia indexes play major roles in the measurement of productivity, price and output change (Jorgenson and Griliches, 1967). They are also closely related to the important Marshallian consumer surplus concept (Rader, 1976; Bruce, 1977; Takayama, 1987). Yet despite their widespread use, the theoretical literature finds that Divisia indexes are valid only under an assumption that is often unrealistic. Hence, a theory of what Divisia indexes measure without this assumption is needed to bridge this disjoint between theory and practice.

In the cost of living index context, the assumption that past authors have found necessary is homotheticity, which means that all Engel curves are straight lines radiating out from the origin.[1] Replaces this assumption with more plausible alternatives yields three main results emerge. First, in the most general case, the Divisia quantity index equals a weighted combination of the standard of living indexes (SLIs) implied by various possible reference price vectors. Second, in special cases where the weights are everywhere non-negative, the Divisia price index equals a product of one cost of living index (CLI) based on the initial utility level and another based on the final utility level. Third, in cases that are still more special, the Divisia price index equals a CLI based on some intermediate utility level. These results can be used to clarify the interpretation of a CLI based on the aggregate demands of a group of consumers.

The organization of this paper is as follows. Section 1 reviews CLI and SLI concepts. Section 2 explains their relationship to Laspeyres and Paasche indexes. Section 3 introduces Divisia indexes and Section 4 discusses their Achilles heel, their dependence on how the researcher specifies the path for prices and income. Section 5 shows how a Divisia price index can equal a CLI based on either the initial or final level of utility. Section 6 discusses conditions under which the Divisia price index equals a CLI evaluated at an intermediate utility level. Sections 7 and 8 show that under less restrictive conditions the Divisia quantity index equals a weighted average of SLIs with reference prices drawn from the path connecting $\mathbf{p}_0$ to $\mathbf{p}_1$ and the Divisia price index equals a product of CLIs based on the initial and final utility levels. Section 9 uses these results to show that a CLI or a chained price index based on aggregate demands summarizes, is at least approximately summarizes, the CLIs of the individuals in the society. Section 10 concludes.

1. Economic Price and Quantity Indexes

Samuelson and Swamy (1974) distinguish between "economic indexes" and Divisia indexes. Economic indexes reflect changes in the expenditure function $e(\mathbf{p}, u)$, which gives the minimum cost of achieving utility u at prices $\mathbf{p}$. The economic price index is also known as a Konüs index or a cost of living index (CLI). It compares the cost of achieving reference utility level u^r at the final price vector $\mathbf{p}_1$ to that cost at the initial price vector $\mathbf{p}_0$:

(1) $$P_E(\mathbf{p}_0, \mathbf{p}_1, u^r) = \frac{e(\mathbf{p}_1, u^r)}{e(\mathbf{p}_0, u^r)}$$

[1] In the productivity context, the stronger assumption of homogeneity of degree 1 is needed to prevent scale economies from being counted as productivity growth.

Fixing prices at a reference price vector $\mathbf{p}^r$ and setting u equal to the utility function $u(\mathbf{q})$, turns the expenditure function into a money metric utility function $e[\mathbf{p}^r, u(\mathbf{q})]$. The economic quantity index, or standard of living index (SLI), uses this function to calculate the relative value of the final consumption bundle $\mathbf{q}_1$ (Samuelson and Swamy,1974, p. 567). It is:

$$Q_E(\mathbf{q}_0, \mathbf{q}_1, \mathbf{p}^r) = \frac{e[\mathbf{p}^r, u(\mathbf{q}_1)]}{e[\mathbf{p}^r, u(\mathbf{q}_0)]} \tag{2}$$

If $\mathbf{q}_0$ and $\mathbf{q}_1$ are optimal consumption bundles given the price-income combinations $(\mathbf{p}_0, Y_0)$ and $(\mathbf{p}_1, Y_1)$, the indirect utility functions $v(\mathbf{p}_0, Y_0)$ and $v(\mathbf{p}_1, Y_1)$ can be substituted for $u(\mathbf{q}_0)$ and $u(\mathbf{q}_1)$ in (2) to obtain a ratio of money metric indirect utility functions. The result is an index that compares the optimal quantities at initial income and prices with the quantities that are optimal at final income and prices. Jorgenson (1990) calculates an index of this type.

Henceforth I will use $P_E(*, u^r)$ as a shorthand for $P_E(\mathbf{p}_0, \mathbf{p}_1, u^r)$ and $Q_E(*, \mathbf{p}^r)$ as a shorthand for $Q_E(\mathbf{q}_0, \mathbf{q}_1, \mathbf{p}^r)$. The first two arguments of $P_E(\cdot)$ or $Q_E(\cdot)$ will be made explicit only when necessary.

2. Laspeyres and Paasche Indexes

Economic indexes depend on the unobservable formula for the expenditure function, but they can be bounded by indexes that depend on observable prices and quantities. Four combinations of observable prices and quantities are possible. They are:

$(\mathbf{p}_1 \cdot \mathbf{q}_0)/(\mathbf{p}_0 \cdot \mathbf{q}_0)$, which is the Laspeyres price index or LPI;

$(\mathbf{p}_1 \cdot \mathbf{q}_1)/(\mathbf{p}_0 \cdot \mathbf{q}_1)$, the Paasche price index or PPI;

$(\mathbf{p}_0 \cdot \mathbf{q}_1)/(\mathbf{p}_0 \cdot \mathbf{q}_0)$, the Laspeyres quantity index or LQI; and

$(\mathbf{p}_1 \cdot \mathbf{q}_1)/(\mathbf{p}_1 \cdot \mathbf{q}_0)$, the Paasche quantity index or PQI.

The Laspeyres indexes are upper bounds for economic indexes and the Paasche indexes are lower bounds. Assuming that $\mathbf{q}_0$ is chosen optimally, $e(\mathbf{p}_0, u_0) = \mathbf{p}_0 \cdot \mathbf{q}_0$, where $u_0 = u(\mathbf{q}_0)$. Also $e(\mathbf{p}_1, u_0) \le \mathbf{p}_1 \cdot \mathbf{q}_0$, since $e(\mathbf{p}_1, u_0)$ is defined as a minimum. Hence the ratio of expenditure functions that equals $P_E(*, u_0)$ is less than or equal to the LPI. Letting $u_1 = u(\mathbf{q}_1)$, similar logic shows that $P_E(*, u_1)$ is greater than or equal to the PPI.

Price index bounds imply quantity index bounds. Dividing the expenditure change $(\mathbf{p}_1 \cdot \mathbf{q}_1)/(\mathbf{p}_0 \cdot \mathbf{q}_0)$ by the LPI yields the PQI, while dividing it by $P_E(*, u_0)$ yields $Q_E(*, \mathbf{p}_1)$. Hence LPI $\ge P_E(*, u_0)$ implies that the PQI $\le Q_E(*, \mathbf{p}_1)$. Similarly, the lower bound property of the PPI implies that the LQI is an upper bound for a standard of living index with reference prices of $\mathbf{p}_0$.

Some weaker bounding conditions do not require specific reference prices or utilities. If a budget of $\mathbf{p}_0 \cdot \mathbf{q}_0$ is sufficient to purchase bundle $\mathbf{q}_1$ at prices $\mathbf{p}_0$, then $\mathbf{q}_0$ is revealed preferred to $\mathbf{q}_1$. Hence, a Laspeyres quantity index less than or equal to 1 implies that utility has not risen:

(3) $$\frac{\mathbf{p}_0\mathbf{q}_1}{\mathbf{p}_0\mathbf{q}_0} \le 1 \Rightarrow u_1 \le u_0$$

Similarly, $\mathbf{q}_1$ is revealed preferred to $\mathbf{q}_0$ if the Paasche quantity index is greater than or equal to 1:

(4) $$\frac{\mathbf{p}_1\mathbf{q}_1}{\mathbf{p}_1\mathbf{q}_0} \ge 1 \Rightarrow u_1 \ge u_0$$

The direction of change of the standard of living index must match the direction of change of utility. Hence, when the LQI is less than or equal to 1, the SLI must be less than or equal to 1 and when the PQI is greater than or equal to 1, the SLI is greater than or equal to 1. For price indexes, this means that: (a) PPI $\ge Y_1/Y_0 \Rightarrow$ CLI $\ge Y_1/Y_0$; and (b) LPI $\le Y_1/Y_0 \Rightarrow$ CLI $\le Y_1/Y_0$.

3. Divisia Indexes

The motivation usually given for Divisia indexes is a desire for consistency between the price and quantity indexes. This is important because deflating an expenditure change to get an implicit quantity index is a common use of price indexes. Economic indexes with the same reference period fail to decompose expenditure changes into mutually consistent price and quantity components. For example, if $u^r = u_0$ and $\mathbf{p}^r = \mathbf{p}_0$ then $Q_E(*, \mathbf{p}^r) \times P_E(*, u^r) =$ $[e(\mathbf{p}_0, u_1) \times e(\mathbf{p}_1, u_0)]/(Y_0)^2$, which is likely to differ from Y_1/Y_0. For the price index times the quantity index to equal Y_1/Y_0, $\mathbf{p}^r$ and u^r must come from different periods. The Paasche and Laspeyres indexes also behave in this way, because LPI $\times$ LQI or PPI $\times$ LQI equals Y_1/Y_0.

In contrast, Divisia indexes do decompose expenditure changes into mutually consistent price and quantity changes. Let $\mathbf{p}_t$: $[0,1] \to \Re^n_{++}$ be a function of t that describes a piecewise smooth path for prices from $\mathbf{p}_0$ to $\mathbf{p}_1$. For example, $\mathbf{p}_t$ might equal $\mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$, a path that traverses the convex hull of $\mathbf{p}_0$ and $\mathbf{p}_1$. Next, let Y_t define a path for income, and let $\mathbf{q}_t$ equal $q(\mathbf{p}_t, Y_t)$, where $q(\mathbf{p}, Y)$ is a function giving the quantity vector that occurs at prices $\mathbf{p}$ and expenditure level Y. Then substituting Y_t for $\mathbf{q}_t \cdot \mathbf{p}_t$ shows that the Laspeyres index for a price change from $\mathbf{p}_t$ to $\mathbf{p}_{t+\Delta t}$ equals $1 + \mathbf{q}_t \cdot (\mathbf{p}_{t+\Delta t} - \mathbf{p}_t)/Y_t$.

A chained Laspeyres price index can be used to evaluate the change from $(\mathbf{p}_0, Y_0)$ to $(\mathbf{p}_1, Y_1)$ by breaking the path into pieces and linking in a new Laspeyres index at each breakpoint. The Divisia price index P_D is defined as the limit of a chained Laspeyres—or, equivalently, Paasche—price index as the linking points grow close enough together to form a continuous line.[2] The limit as Δt approaches zero of the chained Laspeyres index is:

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[2] Similarly, the limit of a continuously chained Hicksian surplus with a reference utility levels $v(\mathbf{p}_t, Y_t)$ is the Marshallian surplus. Thus, the choice between a Hicksian and a Marshallian surplus measure amounts to a choice of between an unchained and a chained measure. Criticisms of the latter measure as conceptually flawed due to a mixing of price and utility changes are, therefore, misleading.

(5)
$$P_D = \lim_{\Delta t \to 0} \prod_{k=1}^{1/\Delta t} \left\{ 1 + \frac{\mathbf{q}_{(k-1)\Delta t} \cdot (\mathbf{p}_{(k)\Delta t} - \mathbf{p}_{(k-1)\Delta t})}{Y_{(k-1)\Delta t}} \right\}$$

Letting $\mathbf{p}_t'$ denote $\partial \mathbf{p}_t/\partial t$ and Y_t' denote $\partial Y_t/\partial t$, $\log[1 + \mathbf{q}_t \cdot (\mathbf{p}_{t+\Delta t} - \mathbf{p}_t)/Y_t]/\Delta t$ has a limit as Δt approaches 0 of $\mathbf{q}_t \cdot \mathbf{p}_t' / Y_t$. Hence, of the logarithm of equation (5) is:

(6)
$$\log P_D = \int_0^1 \frac{\mathbf{q}_t \cdot \mathbf{p}_t'}{Y_t} \, dt \, .$$

A convenient way to express the integrand in equation (6) is as an expenditure-share weighted average of rates of price change. Let $w_{it} = p_{it} q_i(\mathbf{p}_t, Y_t)/Y_t$, the expenditure share for good i. Then $q_i(\mathbf{p}_t, Y_t)\, p_{it}'/Y_t = w_{it}\, p_{it}'/p_{it}$ and $\mathbf{q}_t \cdot \mathbf{p}_t'/Y_t = \mathbf{w}_t \cdot \dot{\mathbf{p}}_t$, where $\mathbf{w}_t$ is the vector of expenditure shares given $(\mathbf{p}_t, Y_t)$ and $\dot{\mathbf{p}}_t$ is a vector of the rates of price change p_{it}'/p_{it}. Equation (6) then becomes:

(7)
$$\log P_D = \int_0^1 \mathbf{w}_t \cdot \dot{\mathbf{p}}_t \, dt \, .$$

The Divisia quantity index, Q_D, resembles the price index. A Laspeyres quantity index from time t to time $t + \Delta t$ equals $1 + \mathbf{p}_t \cdot [\mathbf{q}_{t+\Delta t} - \mathbf{q}_t]/Y_t$. Also, $Y_{t+\Delta t} - Y_t = \mathbf{p}_t \cdot [\mathbf{q}_{t+\Delta t} - \mathbf{q}_t] + \mathbf{q}_t \cdot [\mathbf{p}_{t+\Delta t} - \mathbf{p}_t] + [\mathbf{q}_{t+\Delta t} - \mathbf{q}_t] \cdot [\mathbf{p}_{t+\Delta t} - \mathbf{p}_t]$. Thus, $\mathbf{p}_t \cdot [\mathbf{q}_{t+\Delta t} - \mathbf{q}_t]/[Y_t \, \Delta t]$ has a limit as Δt approaches zero of $\dot{Y}_t - w(\mathbf{p}_t, Y_t) \cdot \dot{\mathbf{p}}_t$, where $\dot{Y}_t$ denotes Y_t'/Y_t. The limit of the logarithm of a sequence of Laspeyres quantity indexes chained at intervals of Δt is, then,

(8)
$$\log Q_D = \int_0^1 \left(\dot{Y}_t - \mathbf{w}_t \cdot \dot{\mathbf{p}}_t \right) dt \, .$$

Equations (7) and (8) are line integrals whose value depends on the path for prices and income and on the demand system specification. A practical way of approximating these theoretical Divisia indexes for empirical applications is to chain at discrete intervals, using average expenditure shares as weights. For example, in addition to observing $\mathbf{p}_0$ and $\mathbf{p}_1$, we might observe prices and quantities at $t = 0.5$. An approximation to equation (7) based on these observations is:

(9)
$$P_D \approx \left[\prod_i (p_{i,0.5}/p_{i0})^{0.5(w_{i0} + w_{i,0.5})} \right] \left[\prod_i (p_{i1}/p_{i,0.5})^{0.5(w_{i,0.5} + w_{i1})} \right] .$$

In empirical contexts, the term “Divisia index” usually means a chained index of this type.

Certain formulas can give an exact value for P_D when the expenditure function has a particular form. For instance, the Törnqvist index formula, which employs simple averages of the initial and final shares, is used twice in equation (9). It gives the exact value of P_D in the case of a homothetic translog expenditure function.[3] Let $p_{it} = p_{i0}(p_{i1}/p_{i0})^t$ for all i. Then w_{it} is

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[3] Theorem 2.16 in Diewert (1976) states that, given a nonhomothetic translog expenditure function, the Törnqvist index equals $P_E(*; u^*)$ when u^* is the geometric mean of u_0 and u_1. Specifying the $\dot{\mathbf{p}}_t$ as constants makes the nonhomothetic translog Divisia price index approximately—but not precisely—equal the Törnqvist index.

a linear function of t because the translog model implies that expenditure shares depend linearly on log prices. Moreover, since $\dot{p}_{it} = \log(p_{i1}/p_{i0})$ is constant, $\mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ is linear in t. Since the integral of a linear function of t equals the change in t (in this case 1) times the average of the endpoint values of the integrand, $P_D = \prod_i (p_{i1}/p_{i0})^{0.5(w_{i0} + w_{i1})}$.

Another noteworthy exact formula for P_D is the Sato-Vartia index. In the CES case, for any i, $w_{it} = \dfrac{w_{i0}(p_{it}/p_{i0})^{1-\sigma}}{\sum_j w_{j0}(p_{jt}/p_{j0})^{1-\sigma}}$. Hence w_{it} is proportional to $w_{i0}(p_{i1}/p_{i0})^{(1-\sigma)t}$ when p_{it} is specified to equal $p_{i0}(p_{i1}/p_{i0})^t$. Substituting this expression for w_{it} into equation (7) and solving the integral gives: $\log P_D = \sum_i \dfrac{\operatorname{logmean}(w_{i0}, w_{i1})}{\sum_j \operatorname{logmean}(w_{j0}, w_{j1})} \log(p_{i1}/p_{i0})$, where $\operatorname{logmean}(w_{j0}, w_{j1})$ equals the logarithmic mean $\dfrac{w_{j1} - w_{j0}}{\log(w_{j1}/w_{jo})}$.

A final example of an exact formula for P_D occurs in the case of Deaton and Muellbauer's (1980) Almost Ideal Demand System (AIDS). When $p_{it} = p_{i0}(p_{i1}/p_{i0})^t$ for all i, this model implies that every w_{it} is a quadratic function of t.[4] Simpson's rule therefore implies that the integral for P_D can be evaluated by putting a two-thirds weight on the expenditure shares implied by the geometric mean of the initial and final values for prices and income. In particular, log P_D = $[\frac{1}{6} w(\mathbf{p}_0, Y_0) + \frac{2}{3} w(\mathbf{p}_{0.5}, Y_{0.5}) + \frac{1}{6} w(\mathbf{p}_1, Y_1)] \cdot \log(\mathbf{p}_1/\mathbf{p}_0)$, where $\log(\mathbf{p}_1/\mathbf{p}_0)$ denotes a vector with elements of the form $\log(p_{i1}/p_{i0})$. This expression also gives a good approximation for log P_D for almost any other demand model when the path is straight line in log price and log income space.

4. THE PATH DEPENDENCE PROBLEM

The advantages of Divisia indexes give them a crucial role in the theories of welfare and productivity measurement. Yet they also a have a problem serious enough to raise questions about their validity: in the absence of the implausible assumption of homotheticity, the value of the Divisia index depends on how the researcher specifies the path that prices and income take in changing from $(\mathbf{p}_0, Y_0)$ to $(\mathbf{p}_1, Y_1)$.[5]

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[4] I am grateful to Rob Feenstra for assistance with the AIDS model Divisia index.

[5] This argument has also been used to dispute the validity of the Marshallian consumer surplus measure: see Silberberg (1972).

Let the demand system be consistent with utility maximization, so that an indirect utility function exists. Rescaling the quantity index integrand by an "integrating factor" equal to $\partial \log v(\mathbf{p}, Y)/\partial \log Y$ gives an expression that integrates to $\log u_1 - \log u_0$ regardless of the path from $(\mathbf{p}_0, Y_0)$ to $(\mathbf{p}_1, Y_1)$. In particular, $v_Y(\mathbf{p}_t, Y_t)(Y_t/u_t)$ times the quantity index integrand $(1/Y_t)[Y'_t - \mathbf{q}_t \cdot \mathbf{p}'_t]\,dt$ equals $(1/u_t)v_Y(\cdot)[Y'_t - \mathbf{q}_t \cdot \mathbf{p}'_t]\,dt$, where $u_t = v(\mathbf{p}_t, Y_t)$. This, in turn, equals $d\log v(\mathbf{p}_t, Y_t)$ since $\mathbf{q}_t = -v_{\mathrm{p}}(\cdot)/v_Y(\cdot)$ by Roy's identity. Hence, utility that is proportional to income to some power implies path independence for the Divisia quantity index. Non-homothetic demand systems lack this property because they make $\partial \log v(\mathbf{p}, Y)/\partial \log Y$ a function of $\mathbf{p}$ and Y.

Hulten (1973) points out that without homotheticity a Divisia index can be made equal to any positive value by cycling around a loop sufficiently often. The path must, therefore, be restricted in some way for a non-homothetic Divisia index to be meaningful. Given utility maximization, an integrating factor factor exists and is globally positive. (See Ville, 1951-2, p. 123.) Hence, considering only paths where the quantity index integrand $\dot{Y}_t - \mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ nowhere changes sign confers enough immunity to the path dependence problem to ensure that the sign of $\log Q_D$ matches the sign of the utility change. An integrand that never reverses its sign implies an integral that either grows monotonically or falls monotonically as the path is traversed. I therefore term paths that have this property "monotonic".

Of course, one would like not just the sign but also the value of the index to be uniquely determined by the points that it is to compare, $(\mathbf{p}_0, Y_0)$ and $(\mathbf{p}_1, Y_1)$. Unfortunately, this is impossible because no particular path can be singled out as the "correct" one except in special circumstances. One such circumstance is when only one price changes. The other is when $(\mathbf{p}_0, Y_0)$ and $(\mathbf{p}_1, Y_1)$ generate demands on the same indifference curve. In this case, the integrand must equal zero everywhere for the path to be monotonic. This means that the path must lie on an indifference curve in price-income space from $(\mathbf{p}_0, Y_0)$ to $(\mathbf{p}_1, Y_1)$.

5. Paths on an Indifference Curve: Divisia Price Indexes as Konüs Indexes

When $(\mathbf{p}_0, Y_0)$ and $(\mathbf{p}_1, Y_1)$ are on distinct indifference surfaces, either of those surfaces can be a route for a monotonic path. A path on the u_0 indifference curve yields a P_D that equals "Laspeyres-Konüs" index $P_E(*, u_0)$, while a path on the u_1 indifference curve yields a P_D equal to a "Paasche-Konüs" index $P_E(*, u_1)$. Figure 1 depicts these paths for a price change from (0.25,2) to (3,0.5) given a constant income of 4 and a utility function of $u = (q_1 - 1)^{0.5}(q_2 - 1)^{0.5}$.

Both paths have two segments. The first segment of Laspeyres-Konüs index path starts at $(\mathbf{p}_0, Y_0)$ and ends at $(\mathbf{p}_1/\lambda_1, Y_0)$, where "$\mathbf{p}_1/\lambda_1$" means that some scalar λ_1 deflates every element of the vector $\mathbf{p}_1$. On this segment Y_t always equals Y_0 and $\mathbf{p}_t$ is such that $v(\mathbf{p}_t, Y_0)$ always equals u_0. Since this segment follows an indifference curve, it has a log price index integral of 0. Also, since $v(\mathbf{p}_1/\lambda_1, Y_0) = u_0$ implies that $v(\mathbf{p}_1, \lambda_1 Y_0) = u_0$, λ_1 must equal $P_E(*, u_0)$.

The next segment of the path runs from $(\mathbf{p}_1/\lambda_1, Y_0)$ to $(\mathbf{p}_1, Y_1)$, which means that prices move along a ray from the origin that passes through $\mathbf{p}_1$. Since every price changes by a factor of λ_1, the price index integral on this segment must equal log λ_1 regardless of how the income path affects the expenditure share weights. Combining the two path segments therefore implies a value of λ_1 for P_D.

To obtain the Paasche-Konüs index, let the first segment of the path have prices start at $\mathbf{p}_0$ and move along a ray to $\mathbf{p}_0\pi_0$, where the scalar π_0 is such that $v(\mathbf{p}_0\pi_0, Y_1) = v(\mathbf{p}_1, Y_1)$. Integrating over this part of the path contributes log π_0 to the value of the log price index. The second segment of the path keeps $v(\mathbf{p}_t, Y_1)$ equal to u_1 until it arrives at $(\mathbf{p}_1, Y_1)$. The combined integral for P_D therefore equals log π_0. Moreover, since $v(\mathbf{p}_0, Y_1/\pi_0) = u_1$, π_0 equals the Paasche-Konüs index.

If the value of Q_D is of no interest, a path where prices move in a smooth fashion from p_0 to p_1 can be used to generate a P_D that equals λ_1 or π_0. On these paths, instead of keeping $\mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ equal to 0, income is adjusted to keep the log quantity index integrand equal to 0. Holding utility constant effectively results in the integration of a Hicksian (or compensated) demand function. This yields a Konüs index because the integral of a Hicksian demand function is an expenditure function. The following propositions consider paths of this type.

PROPOSITION 1: *The Divisia price index evaluated with income path* $Y_t = e(\mathbf{p}_t, u_0)$ *equals the Laspeyres-Konüs index* $P_E(*, u_0)$.

PROOF: By the definition of $P_E(*, u_0)$,

$$\log P_E(*, u_0) = \int_0^1 \frac{\partial \log e(\mathbf{p}_t, u_0)}{\partial t} dt \tag{10}$$

Shepard's lemma implies that $q[\mathbf{p}_t, e(\mathbf{p}_t, u_0)]$ equals $\partial e(\mathbf{p}, u_0)/\partial \mathbf{p}$ evaluated at $\mathbf{p} = \mathbf{p}_t$. Consequently,

$$\frac{\partial \log e(\mathbf{p}_t, u_0)}{\partial t} = \frac{q[\mathbf{p}_t, e(\mathbf{p}_t, u_0)] \cdot \mathbf{p}'_t}{e(\mathbf{p}_t, u_0)} = w[\mathbf{p}_t, e(\mathbf{p}_t, u_0)]\dot{\mathbf{p}}_t \tag{11}$$

Equation (7) implies that the log Divisia price index evaluated with the income path in question equals the integral of the right side of equation (11). *QED*

The next proposition states that the Divisia index equals $P_E(*, u_1)$ when the consumer's income path keeps utility at u_1 as relative prices change.

PROPOSITION 2: *The Divisia price index evaluated with* $Y_t = e(\mathbf{p}_t, u_1)$ *equals the Paasche-Konüs index* $P_E(*, u_1)$ *for the price change from* $\mathbf{p}_0$ *to* $\mathbf{p}_1$.

PROOF: Analogous to the proof of Proposition 1.

6. DIVISIA INDEXES AS INTERMEDIATE UTILITY CLIS WHEN ENGEL CURVES ARE GLOBALLY CONVEX OR GLOBALLY CONCAVE

A Divisia price index based on a monotonic path evaluates the price change using utility levels that vary from u_0 to u_1. One might, therefore, suppose that a path that holds income constant and allows only one price to change would nearly always imply an intermediate utility CLI interpretation for the Divisia index. The following two examples show that even simple departures from homotheticity can prevent this from happening. Since luxury status is presumably more common at high prices, the example that has P_D below the lower CLI may be more representative of the usual state of affairs. Calculation details for the examples are relegated to the appendix.

For the first example, let the indirect utility function be $(Y - p_1 - p_2)(p_1)^{-0.5}(p_2)^{-0.5}$, which arises from the utility function underlying Figure 1. Let p_1 rise from 0.5 to 1.5 with Y constant at 3. Then $P_E(*, u_0)$ equals 1.699, $P_E(*, u_1)$ equals 1.677, but P_D equals 1.704. Figure 2 shows why both CLIs give less weight to the price increase than P_D. It shows the compensated integrand of Proposition 1, $q_1[\mathbf{p}, e(\mathbf{p}, u_0)]/e(\mathbf{p}, u_0)$, as the line that starts out highest and the compensated integrand of Proposition 2, $q_1[\mathbf{p}, e(\mathbf{p}, u_1)]/e(\mathbf{p}, u_1)$, as the line that starts out lowest. These lines cross at $p_1 = 1$ because good 1 is a luxury below this price and a necessity above it. The ordinary integrand $q_1(\mathbf{p}, Y)/Y$ is always between the compensated integrands, but as utility approaches u_1 it switches from being close to the u_0 compensated integrand to being close to the u_1 integrand. It therefore integrates to a higher value than either of the functions that bracket it.

The second example comes from Hausman's (1981) paper on Hicksian consumer surplus measures for linear demand curves. Hausman's gasoline example has $q = -14.22p + 0.082Y + 4.95$, where $Y = \$720$ per month. The \$0.75 to \$1.50 price change that Hausman considers implies a path entirely in the luxury range. Hence P_D, which equals 1.0513, is between $P_E(*, u_1) = 1.0510$, and $P_E(*, u_0) = 1.0516$. On the other hand, below 34.8¢ per gallon gasoline becomes a necessity. A price change from 20¢/gallon to 50¢/gallon therefore implies a P_D below both CLIs. In particular, $P_D = 1.02489$ and $P_E(*, u_0) = P_E(*, u_1) = 1.02490$. Hence no u^* exists in $[u_0, u_1]$ such that $P_E(*, u^*) = P_D$.

Most non-homothetic demand systems have a region where a good has concave Engel curves, which make it a necessity, and another region where that same good has convex Engel curves, which make it a luxury. Movements from one region to the other cause a reversal in the order of the u_0 and u_1 compensated integrands. In the above examples, this allows the ordinary Divisia integrand $\mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ to integrate to a value outside of the range of $P_E(*, u)$ for $u \in [u_0, u_1]$. On the other hand, the following proposition states that all the Engel curves bending the same way is a sufficient condition for P_D to equal an intermediate utility CLI.

PROPOSITION 3: *Let only one price change, and let the affected good not change status between luxury and strict necessity anywhere in the region bounded by the segments of the u_0 and u_1 indifference curves that surround the path. Then* $P_E(*, u^*) = P_D$ for some $u^* \in [u_0, u_1]$.

PROOF: Case (*i*): $(p_{i1} - p_{i0})[\partial w_i(\mathbf{p}_t, Y_t)/\partial Y_t] \geq 0 \; \forall \; t \in [0,1]$, where good i's price is assumed to be the one that changes. Either good i is a luxury with a rising price or a necessity with a falling price. If it is a luxury then it has a smaller budget share at lower utility levels and $u_1 < u_0$. If good i is a necessity, then it has a larger budget share at lower utility levels and $u_0 < u_1$.

Let $\mathbf{w}_{0t}$ denote $w[\mathbf{p}_t, e(\mathbf{p}_t, u_0)]$ and $\mathbf{w}_{1t}$ denote $w[\mathbf{p}_t, e(\mathbf{p}_t, u_1)]$. Then $\mathbf{w}_{1t} \cdot \dot{\mathbf{p}}_t \leq \mathbf{w}_t \cdot \dot{\mathbf{p}}_t \leq \mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t$ $\forall \; t \in [0,1]$. Proposition 1 says that $\mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t$ integrates to log $P_E(*, u_0)$, and Proposition 2 says that $\mathbf{w}_{1t} \cdot \dot{\mathbf{p}}_t$ integrates to log $P_E(*, u_1)$. Hence, integrating the three terms in the inequality gives: log $P_E(*, u_1) \leq$ log $P_D \leq$ log $P_E(*, u_0)$. By continuity of $P_E(*, u)$ in u, this implies the existence of a u^* in $[u_0, u_1]$ such that $P_E(*, u^*) = P_D$..

Case (*ii*): $(p_{i1} - p_{i0})[\partial w_i(\mathbf{p}_t, Y_t)/\partial Y_t] \leq 0 \; \forall \; t \in [0,1]$. Either good i is a necessity with a rising price or a luxury with a falling price. In this case, $\mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t \leq \mathbf{w}_t \cdot \dot{\mathbf{p}}_t \leq \mathbf{w}_{1t} \cdot \dot{\mathbf{p}}_t \; \forall \; t \in [0,1]$. Integrating then shows that $P_E(*, u_0) \leq P_D \leq P_E(*, u_1)$. Continuity again implies the existence of a u^* such that $P_E(*, u^*) = P_D$. *QED*

The globally identical curvature condition of Proposition 3 is satisfied by two important models. They are the non-homothetic translog model and Deaton and Muellbauer's (1980) AIDS model. Indeed, for these models, any path that follows a straight line in log price space and keeps u_t in between u_0 and u_1 will generate a P_D with an intermediate utility CLI interpretation. These models imply that $\partial \mathbf{w}_t/\partial Y_t$ is proportional to vector of constants. This precludes reversals in the order of $\mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t$ and $\mathbf{w}_{1t} \cdot \dot{\mathbf{p}}_t$ if $\dot{\mathbf{p}}_t$ is a vector of constants.

7. DIVISIA QUANTITY INDEXES AS WEIGHTED AVERAGES OF SLIS WHEN ENGEL CURVES HAVE UNIFORM CURVATURE

Goods can be luxuries at some prices and necessities at others without affecting the interpretation of a Divisia quantity index as a weighted average of economic indexes that have reference prices drawn from the path. A weighted average interpretation for the quantity index requires only that regions of convex curvature and regions of concave curvature not occur on the *same* Engel curve. In a paper on revealed preference theory, Freixas and Mas-Colell (1987) use the term "uniform curvature" for this condition.

Assuming that $u_1 \neq u_0$, the weighting function that generates this average is the derivative of a function F_t that measures the relative distance of the ordinary price index integrand $\mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ from the utility u_0 compensated integrand $\mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t$. This generally means that points on the path where utility is changing fastest have the highest weights, while points where the path is moving along an indifference curve have zero weights. The path dependence problem is thus a reflection of the fact that different paths imply different weighting schemes for combining the values of $Q_E(*, \mathbf{p}_t)$.

Let $\dot{\delta}_t = \mathbf{w}_t \cdot \dot{\mathbf{p}}_t$ and let $\dot{\lambda}_t = \mathbf{w}_{0t} \cdot \dot{\mathbf{p}}_t$, which is the time derivative of the logarithm of $\lambda_t = P_E(\mathbf{p}_0, \mathbf{p}_t, u_0) = e(\mathbf{p}_t, u_0)/Y_0$. Similarly, let $\dot{\pi}_t = -\mathbf{w}_{1t} \cdot \dot{\mathbf{p}}_t$, where $\pi_t = P_E(\mathbf{p}_t, \mathbf{p}_1, u_1)$, or $Y_1/e(\mathbf{p}_t, u_1)$. Finally, define F_t as a continuously differentiable monotonic function from [0,1] to [0,1] that solves the equation $\dot{\delta}_t = (F_t)(-\dot{\pi}_t) + (1 - F_t)\dot{\lambda}_t$. At points where $\dot{\lambda}_t \neq -\dot{\pi}_t$,

(12) $$F_t = \frac{\dot{\delta}_t - \dot{\lambda}_t}{-\dot{\pi}_t - \dot{\lambda}_t} = \frac{(\mathbf{w}_t - \mathbf{w}_{0t}) \cdot \dot{\mathbf{p}}_t}{(\mathbf{w}_{1t} - \mathbf{w}_{0t}) \cdot \dot{\mathbf{p}}_t}$$

At points where $\dot{\lambda}_t = -\dot{\pi}_t$, to preserve continuity F_t must equal the limit of $[\dot{\delta}_\tau - \dot{\lambda}_\tau]/[\dot{\pi}_\tau - \dot{\lambda}_\tau]$ as τ approaches t if this limit exists. (This limit will exist if $\dot{\lambda}_t \neq -\dot{\pi}_t$ almost everywhere.) If $\dot{\lambda}_t$ equals $-\dot{\pi}_t$ everywhere, as it would in the homothetic case or if all prices changed by the same proportion, let $F_t = t$.

For an important class of demand systems that includes the AIDS and non-homothetic translog models, the appendix shows F_t is simply the relative change in utility when the path reaches the point $(\mathbf{p}_t, Y_t)$:

(13) $$F_t = \frac{u_t - u_0}{u_1 - u_0}$$

The class of demand systems that yield equation (13) is known as the "PIGLOG" class. Muellbauer (1975) uses this term for models where the element of $w(\cdot)$ for any good i has the form

(14) $$w_i(\mathbf{p}, Y) = g(\mathbf{p}, Y)\, a_i(\mathbf{p}) + b_i(\mathbf{p})$$

with $g(\mathbf{p}, Y) = \log Y$.

The only other case where prices have no effect on $g(\cdot)$ has $g(\cdot) = Y^\varepsilon$. Muellbauer terms this "PIGL" (for "price independent generalized linear"). In the PIGL case, F_t equals the relative change in a monotonic transformation of a money metric utility function:

(15) $$F_t = \frac{e(\mathbf{p}_t, u_t)^\varepsilon - e(\mathbf{p}_t, u_0)^\varepsilon}{e(\mathbf{p}_t, u_1)^\varepsilon - e(\mathbf{p}_t, u_0)^\varepsilon}$$

As an illustration of the calculation of F_t, let $\mathbf{p}_t = (2^t, 2^{-t})$ connect $\mathbf{p}_0 = (1,1)$ and $\mathbf{p}_1 = (2, 0.5)$, with Y constant at 4. Also, let the demand functions be: $q_1 = 0.5 + 0.5(Y - p_2)/p_1$; and $q_2 = 0.5 + 0.5(Y - p_1)/p_2$. These demands arise from the indirect utility function $(Y - p_1 - p_2)(p_1)^{-0.5}(p_2)^{-0.5}$, so $u_0 = 2$ and $u_1 = 1.5$. Solving for $e(\mathbf{p}_t, u_0)/e(\mathbf{p}_0, u_0)$ shows that $\lambda_t = (2 + 2^t + 2^{-t})/(2 + 1 + 1)$. Consequently $\dot{\lambda}_t = (\log 2)(2^t - 2^{-t})/(2 + 2^t + 2^{-t})$. Similarly, $\pi_t = (1.5 + 2 + 0.5)/(1.5 + 2^t + 2^{-t})$, and $-\dot{\pi}_t = (\log 2)(2^t - 2^{-t})/(1.5 + 2^t + 2^{-t})$. Finally, an expression for $\dot{\delta}_t$ is:

(16)

$$\mathbf{w}_t \cdot \dot{\mathbf{p}}_t = \left[\frac{1}{2} + \frac{1}{2Y}\{P_1(t) - P_2(t)\}\right](\log 2) + \left[\frac{1}{2} + \frac{1}{2Y}\{P_2(t) - P_1(t)\}\right](-\log 2) = \frac{1}{4}\{2^t - 2^{-t}\}(\log 2)$$

Substituting the expressions for $\dot{\lambda}_t$, $-\dot{\pi}_t$ and $\dot{\delta}_t$ into equation (12) gives:

$$F_t = \frac{4^{-1}(2 + 2^t + 2^{-t})^{-1}}{(1.5 + 2^t + 2^{-t})^{-1} - (2 + 2^t + 2^{-t})^{-1}} \quad . \tag{17}$$

Incidentally, for this example $P_E(*, u_0) = 1.125$, $P_D =$ exp(0.125), or 1.133, and $P_E(*, u_1) = 1.143$.

The next proposition expresses $\log Q_D$ using the derivative of F_t to weight the logarithmic SLIs that have a $\mathbf{p}^r$ drawn from the price path $\{\mathbf{p}_t;\ 0 \le t \le 1\}$. If all the weights are non-negative the expression defines a weighted average of the SLIs.

PROPOSITION 4: *Assuming $u_1 \neq u_0$, let f_t be the derivative of a function F_t that solves the equation $\dot{\delta}_t = F_t(-\dot{\pi}_t) + [1 - F_t]\dot{\lambda}_t$, with $F(0) = 0$ and $F(1) = 1$. Then*

$$\log Q_D = \int_0^1 \log\left[\frac{e(\mathbf{p}_t, u_1)}{e(\mathbf{p}_t, u_0)}\right] f_t\, dt \quad . \tag{18}$$

The Divisia quantity index uses the weighting function f_t to combine all the standard of living indexes that have a reference price vector of $\mathbf{p}_t$ for a $t \in [0,1]$.

PROOF: From equation (8),

$$\log Q_D = \log(Y_1/Y_0) - \int_0^1 \dot{\delta}_t\, dt . \tag{19}$$

Substituting $-F_t\dot{\pi}_t + [1 - F_t]\dot{\lambda}_t$ for $\dot{\delta}_t$ gives:

$$\log P_D = \int_0^1 \{-F_t\dot{\pi}_t + [1 - F_t]\dot{\lambda}_t\}\, dt . \tag{20}$$

Now integrate the first term inside the brackets in equation (20) by parts:

$$\int_0^1 -F_t\dot{\pi}_t\, dt = -F_t[\log \pi_t]\Big|_0^1 + \int_0^1 \log \pi_t\, f_t\, dt$$

$$= \int_0^1 \log \pi_t\, f_t\, dt . \tag{21}$$

Next integrate the second term inside the brackets in equation (20) by parts:

$$\int_0^1 [1 - F_t]\,\dot{\lambda}_t\,dt = [1 - F_t][\log \lambda_t]\Big|_0^1 + \int_0^1 \log \lambda_t\, f_t\, dt$$

$$(22) \qquad = \int_0^1 \log \lambda_t\, f_t\, dt\,.$$

Substituting for the two components of the integral in equation (20) now gives:

(23)

$$\int_0^1 \dot{\delta} dt = \int_0^1 [\log \pi_t + \log \lambda_t] f_t\, dt = \int_0^1 \log\left[\frac{Y_1}{e(\mathbf{p}_t, u_1)} \frac{e(\mathbf{p}_t, u_0)}{Y_0}\right] f_t\, dt = \log(Y_1 / Y_0) + \int_0^1 \log\left[\frac{e(\mathbf{p}_t, u_0)}{e(\mathbf{p}_t, u_1)}\right] f_t\, dt$$

Substituting into equation (19) then establishes the desired result.

QED

Proposition 4 shows that the Divisia quantity index has a satisfactory interpretation as an average SLI if F_t is monotonic. For PIGLOG demands, F_t is monotonic if and only if u_t is monotonic. For PIGL demands, equation (15) shows that changes in u_t along the path have a first order effect on F_t but changes in $\mathbf{p}_t$ have a second order effect that depends on a cross-partial derivative with respect to u and $\mathbf{p}$. This means that F_t will be monotonic if the path makes u_t strongly monotonic, but F_t can be non-monotonic if the path nearly parallels an indifference curve. In the latter case, however, the magnitude of the distortion from the negative weights on some of the log $Q_E(*, \mathbf{p}_t)$ will be slight because $\log Q_E(*, \mathbf{p}^r) \approx 0$ regardless of the value of $\mathbf{p}^r$ when $u_0 \approx u_1$. Moreover, we know that any monotonic path will give a result on the correct side of 0.

PIGL and PIGLOG demand systems are special cases of the kind of individual demand systems that cause the Weak Axiom of Revealed Preference (WARP) to hold in the aggregate given an income distribution that is independent of prices. In particular, Freixas and Mas-Colell (1987) find that the two conditions for WARP to hold in the aggregate are: (a) the expenditure shares have the functional form of equation (14)[6]; and (b) Engel curves extended by straight lines from $q(\mathbf{p}, Y^{min})$ to the origin do not include both strictly convex and strictly concave regions.[7] If $g(\mathbf{p}, Y)$ is C^3, condition (b) implies that $g(\mathbf{p}, Y)$ is monotonic in Y for all $Y \in (Y^{min}, Y^{max})$. As in equation (15), F_t then equals the relative change in a monotonic transformation of a money metric utility function. Hence, Freixas and Mas-Collel's conditions have similar implications to PIGL. A strongly monotonic u_t guarantees a monotonic F_t, but a path that almost runs along an indifference curve may not.

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[6] Lewbel (1991) classifies demand systems based on the rank of the space spanned by their Engel curves. Equation (14) implies that the demand system is rank 2. Homothetic demand systems are rank 1.

[7] Another way of stating this second condition is that $\frac{\partial^2 q_i(\mathbf{p}, Y)}{\partial^2 Y}$ cannot change sign at any $Y \in (Y^{min}, Y^{max})$ or have the opposite sign from $\frac{\partial w_i(\mathbf{p}, Y)}{\partial Y}$ evaluated at Y^{min}.

8. P_D AS A COMBINATION OF LASPEYRES-KONÜS AND PAASCHE-KONÜS INDEXES

If Q_D has a weighted average interpretation, continuity of $Q_E(*, \mathbf{p}_t)$ in t implies the existence of a $t^* \in [0,1]$ such that $Q_E(*, \mathbf{p}_{t^*}) = Q_D$. Duality of price and quantity indexes then implies that $\lambda_{t^*} \times \pi_{t^*} = P_D$. Thus under the weak conditions that the path is monotonic and does not pass through a region where the Engel curves have inflection points, the Divisia price index evaluates part of the price change from $\mathbf{p}_0$ to $\mathbf{p}_1$ at utility level u_0 and the rest of the change at utility level u_1. The next proposition formalizes these notions.

PROPOSITION 5: *Let F_t be monotonic. Then: (a) there exists a $t^* \in [0,1]$ such that an SLI with reference price vector $\mathbf{p}_{t^*}$ equals the Divisia quantity index; and (b) chaining a Laspeyres-Konüs index for the change from* $\mathbf{p}_0$ to $\mathbf{p}_{t^*}$ *and a Paasche-Konüs index for the change from $\mathbf{p}_{t^*}$ to $\mathbf{p}_1$ gives the Divisia price index.*

PROOF: By Proposition 4, a monotonic F_t implies that $\log Q_D$ is a weighted average of the $\log Q_E(*, \mathbf{p}_t)$. It must therefore be the case that:

$$\min_{t \in [0,1]} \left[Q_E(*, \mathbf{p}_t)\right] \le Q_D \le \max_{t \in [0,1]} \left[Q_E(*, \mathbf{p}_t)\right]. \tag{24}$$

Since the expenditure function is continuous in prices, $Q_E(*, \mathbf{p})$ is a continuous in prices. Moreover, by assumption, $\mathbf{p}_t$ is continuous in t. Therefore $Q_E(*, \mathbf{p}_t)$ is a continuous function of t. Inequality (23) then implies existence of a $t^* \in [0,1]$ such that $Q_E(*, \mathbf{p}_{t^*}) = Q_D$.

Part (b) of Proposition 5 follows from the fact that $\log P_D = \log(Y_1/Y_0) - \log Q_D = \log \lambda_{t^*} + \log \pi_{t^*}$. *QED*

When F_t is not monotonic over the entire range of t, the path can be broken at any point where f_t changes sign to obtain three or more segments that are internally monotonic. Path without undulations or loops rarely have more than one region where F_t declines. Let t_1 and t_2 be the bounds of such a region, meaning that $f_t \ge 0$ for $0 \le t \le t_1$, $f_t < 0$ for $t_1 < t < t_2$, and $f_t \ge 0$ for $t_2 \le t \le 1$. Then there exist $t_1^* \in [0, t_1]$, $t_2^* \in (t_1, t_2)$ and $t_3^* \in [t_2, 1]$ such that:

$$\begin{aligned} \log Q_D = {} & F_{t_1} \log Q_E(*, \mathbf{p}_{t_1^*})] + [F_{t_2} - F_{t_1}] \log Q_E(*, \mathbf{p}_{t_2^*}) \\ & + [1 - F_{t_2}] \log Q_E(*, \mathbf{p}_{t_3^*}). \end{aligned} \tag{25}$$

Equation (25) implies that the Divisia quantity index equals the product of three SLIs raised to powers that sum to 1. The first of these values the change from u_0 to u_1 based on a reference price from the first segment of the path, the second takes its reference price from the second segment of the path and has a negative power, and the third takes its reference price from the region where F_t resumes rising. If the index with a negative exponent is far outside the range of the indexes with positive exponents, Q_D may differ from every $Q_E(*, \mathbf{p}_t)$.

9. An Application to the "Representative Consumer" Problem

Official Consumer Price Indexes attempt to summarize the cost of living changes of large groups of consumers with diverse income levels and preference structures. A practical solution to the group CLI problem is to calculate a cost of living index for a "representative consumer" whose expenditure patterns mimic those of the group as a whole. Despite the common sense appeal of this procedure, theorists have viewed it as unjustifiable (Kirman, 1992; Lewbel, 1994, p. 1834).

Pollak (1981) offers two theoretical solutions to problem of how to aggregate over consumers. The "social cost of living index" uses a social welfare function to calculate the cost of attaining a reference level of social welfare at $\mathbf{p}_0$ and $\mathbf{p}_1$. Jorgenson (1990) fleshes out this proposal and implements it empirically.

The second possible solution is based on Scitovsky contours, which hold constant the cost to a hypothetical social planner of keeping every consumer in the group at a reference utility level. For example, Pollak's (1980, p. 275;1981, p. 328) "Scitovsky-Laspeyres" index equals the relative cost at prices $\mathbf{p}_1$ of keeping all consumers at the utility level they had when prices equaled $\mathbf{p}_0$. The ordinary Laspeyres index is an upper bound for the Scitovsky-Laspeyres index. Also, the Paasche index is a lower bound for a Scitovsky-Paasche index, which uses consumers' period 1 utilities.

Consider, for example, a society of two consumers with identical constant incomes of Y. Let consumer A buy only good 1 and consumer B buy only good 2. Since the aggregate expenditure shares are constant, the society has a representative consumer who has Cobb-Douglas preferences and a CLI that equals the geometric mean of price relatives, or $\sqrt{2 \times 0.5}$. Now let $\mathbf{p}_0$ = [1 2] and $\mathbf{p}_1$ = [2 1]. To have the same utility as at time 0, consumer A must receive Y units of good 1 at time 1 and B must receive $Y/2$ units. The Scitovsky-Laspeyres index therefore equals $[2Y + (Y/2)]/2Y$, or 1.25. The Scitovsky-Laspeyres index is an arithmetic mean of individual consumers' CLIs. Also, the Scitovsky-Paasche index is a harmonic mean of individuals' CLIs, or 0.80.

An important axiomatic property for index numbers is the time-reversal test, which requires consistent treatment of the reverse of the original price change. This means that the Scitovsky-Laspeyres index should value the reverse change from [2 1] to [1 2] as a 20 percent decrease. Instead, it deems the reverse change to be as much of an increase as the forward change. In view of the well-documented tendency of arithmetic means to cause trouble in index number contexts, the poor axiomatic properties of Scitovsky-Laspeyres index are not surprising.

One way to pass the time reversal test is to use a Fisher index, which equals a geometric average of the Laspeyres and Paasche indexes. Diewert (1983, p. 191) shows that a group price index in between the Paasche and Laspeyres indexes tracks the cost of staying on a Scitovsky contour that uses some set of intermediate utility levels. Thus, Scitovsky contours can furnish a conceptual underpinning for using the Fisher index as a measure of a group CLI.

Nevertheless, the cost of living index formula for the representative consumer—when one exists—has even better axiomatic properties than the Fisher index. In particular, the CLI formula satisfies transitivity tests when it is chained that the Fisher index will generally not satisfy. Indeed, a Scitovsky contour argument cannot be used to justify a chained index. Moreover, the CLI formula may be simpler than the Fisher formula. Given the constant expenditure shares of the above example, the Fisher index equals a geometric mean of identically weighted arithmetic and harmonic means of the price relatives. Applying the geometric mean to the price relatives themselves seems more natural and straightforward than this complicated procedure.

To avoid complicated questions that are tangential to the concerns of the present paper, I assume that consumers have constant shares of aggregate income. These questions arise because, if the income distribution varies with the price structure, aggregate demands may reflect producer behavior rather than consumer behavior.[8]

If individuals have constant shares of aggregate income and individual preferences are homothetic but not necessarily identical, the representative consumer's CLI has an interpretation as a social cost of living index of Pollak's first type. The social welfare function that underlies this social CLI is a geometric mean that weights individuals' utilities in proportion their incomes.[9] In addition, Blackorby and Russell (1993) have shown that giving consumers fixed proportions of aggregate income can maximize a social welfare function if consumers have certain types of PIGLOG or PIGL expenditure functions that need not be identical. This implies existence of a representative consumer who also has PIGLOG or PIGL preferences. The CLI of this representative consumer again has an interpretation as a social cost of living index.

The formula that furnishes the representative consumer's CLI has an important alternative interpretation. As a social Divisia index, this formula is as an average of Divisia indexes of the members of the society. The market-level Marshallian demand function of a group of consumers who face common prices is the sum of the individual consumers' Marshallian demand functions. Since Divisia indexes are based on Marshallian demand functions, the logarithmic Divisia index implied by market-level demands equals a weighted average of individual consumers' log Divisia indexes, where the weights are consumers' shares of aggregate income if these are constant.

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8 In an endowment economy, a redistribution of income and a reversal of this redistribution will generally both generate positive sums of compensating variations because in this circumstance equilibrium prices and quantities reflect producers' substitution between outputs rather than consumers' substitution between inputs. This phenomenon is known as the Boadway paradox.

9 To be precise, the individual utility functions must be linear homogeneous for this to work, but any homothetic utility function is observationally equivalent to a linear homogeneous one except in situations that reveal the degree of risk aversion. This follows from results in Eisenberg (1961) or Jerison (1994). Representative consumer results on aggregating discrete choice models in Anderson, de Palma and Thisse (1992) and Feenstra (1995) are special cases of this.

When individual preferences are neither homothetic nor of the types considered by Blackorby and Russell, a representative consumer will fail to exist or exist but violate the Pareto principle (Jerison, 1994.) In the former case, a representative consumer's CLI is, of course, unavailable, but social Divisia indexes still exist. The major difficulty with these indexes is that, with three or more consumers and goods, a path that is a closed loop may have a positive integrand for log Q_D at every point. Although this implies some inconsistency between the indifference curves of different subspaces, as long as loops are avoided the practical magnitude of the distortion is usually slight.

The possibility of violations of the Pareto principle is, however, the main source of theorists' opposition to the use of the representative consumer for normative purposes. A violation of the Pareto principle occurs when the aggregate demands imply a positive log Q_D while every consumer suffers a welfare loss. Violations of the Pareto principle are possible either with or without a representative consumer because a path that keeps the integrand for the aggregate log Q_D positive at every point may imply integrands for individual consumers that change sign. In particular, although some consumer must have a positive quantity index integrand whenever the aggregate integrand is positive, the identity of the consumers with positive integrands may change from point to point. When individual consumers all have non-monotonic paths, they can all have a log Q_D that has the wrong sign.

Nevertheless, micro level consumption data are so often unavailable or impractical to use that if applied economists are to be forbidden from basing welfare judgments on aggregate data, our discipline will be of little value in many practical situations. Moreover, however disturbing violations of the Pareto principle may be, they do not imply the presence of large distortions in the aggregate Divisia price and quantity indexes. Pareto principle violations occur when the path is nearly parallel to individual consumers' indifference curves. In this circumstance, the magnitude of the distortion will be small in the absence of loops because every reference price vector $\mathbf{p}_t$ will yield an SLI with an absolute value near 0. Furthermore, in the more usual circumstance of a path with a strongly monotonic effect on welfare, every consumer will have a log Q_D that is a proper average of valid SLIs. In particular, when Laspeyres or Törnqvist index formula based on aggregate data is chained at points that do not involve price oscillations or loops, the result will generally be a reasonable summary statistic for individual consumers' cost of living or standard of living changes.

10. CONCLUSION

Samuelson and Swamy (1974, p. 592) conclude by suggesting that "one must not expect to be able to make the naïve measurements that untutored common sense always longs for." Here I have attempted to show that the non-homothetic Divisia index, which seems to be the subject of this remark, is more justifiable than has been supposed. This is fortunate, since the non-homothetic Divisia index often provides the most realistic description of the theoretical concept that empirical index numbers tend towards.

The problem with non-homothetic Divisia indexes is their dependence on the researcher's choice of a path. Here I have shown that different paths correspond to different weighting schemes for possible reference price vectors in the SLI. Points on the path where utility is changing fastest in a forward direction generally have the highest weights, and points where utility is moving backwards generally have negative weights. A path where prices and income move monotonically towards their final configuration is unlikely to include a significant region of negative weights. The Divisia quantity index is then a weighted average of the SLIs that use the points in between $\mathbf{p}_0$ and $\mathbf{p}_1$ as reference price vectors. Also, the Divisia price index effectively uses a Laspeyres-Konüs index to value part of the price change and a Paasche-Konüs index to value the rest of it.

In numerical examples, all paths that avoid swerves and loops invariably yield very similar values for the Divisia index. This is not surprising, since any such path is likely to provide a reasonable welfare change measure. A practical caveat for users of chained indexes is, however, that paths containing kinks or loops may not yield an index with a sensible interpretation. In particular, links that coincide with a back and forth motion of prices can, over time, greatly distort a chained index. Consequently, chaining more often than annually should be avoided when prices have seasonal fluctuations. Linking should instead be timed to occur when the structure of relative prices is as close to its starting point as possible.

A Divisia index can be used to investigate the meaning of chained or Törnqvist indexes based on market-level demand curves because these demand curves imply a log Divisia index that is a weighed average of individual consumers' log Divisia indexes. The Divisia price index calculated from aggregate data is usually an average or summary of individual consumers' CLIs unless the path contains kinks or loops. Under circumstances where it is not an average CLI, the distortion will be small.

This is not to deny that a detailed examination of the distribution of individuals' CLIs would yield valuable information obscured by a single, aggregate statistic. The point is simply that the practical, common-sense procedures based on aggregate data are more justifiable than the theorists have supposed.

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Appendix

Example of a Divisia Index for a Stone-Geary Demand System: With two goods, the Stone-Geary indirect utility function is: $v(\mathbf{p},Y) = (Y - p_1\gamma_1 - p_2\gamma_2)(p_1)^{-\beta_1}(p_2)^{-(1-\beta_1)}$. The expenditure share of good 1 then equals:

$$w_1(\mathbf{p},Y) = \beta_1(1 - p_2\gamma_2/Y) + (1 - \beta_1)(\gamma_1/Y)\, p_1. \tag{A-1}$$

Let p_1 change from $p_{1,0}$ to $p_{1,1}$ while all other variables remain constant. Solving for the Divisia price index gives:

$$\log D = \beta_1(1 - p_2\gamma_2/Y)\log(p_{1,1}/p_{1,0}) + (1 - \beta_1)\gamma_1(p_{1,1} - p_{1,0})/Y. \tag{A-2}$$

Evaluating $e(\mathbf{p}_1, u_0)$ gives:

$$p_{1,1}\gamma_1 + p_2\gamma_2 + u_0(p_{1,1})^{\beta_1}(p_2)^{(1-\beta_1)} = p_{1,1}\gamma_1 + p_2\gamma_2 + (Y - p_{1,0}\gamma_1 - p_2\gamma_2)(p_{1,1}/p_{1,0})^{\beta_1}. \tag{A-3}$$

The Laspeyres-Konüs index is, therefore,

$$L = \frac{p_{1,1}\gamma_1 + p_2\gamma_2 + (Y - p_{1,0}\gamma_1 + p_2\gamma_2)(p_{1,1}/p_{1,0})^{\beta_1}}{Y} \tag{A-4}$$

Similarly, the Paasche-Konüs index is,

$$\Pi = \frac{Y}{p_{1,0}\gamma_1 + p_2\gamma_2 + (Y - p_{1,1}\gamma_1 + p_2\gamma_2)(p_{1,1}/p_{1,0})^{-\beta_1}} \tag{A-5}$$

Let $\gamma_1 = \gamma_2 = p_2 = 1$ and let $\beta_1 = 0.5$. Good 1 switches status from luxury to necessity when p_1 equals $\frac{\beta_1}{1-\beta_1}\frac{\gamma_2}{\gamma_1}p_2$. The value of this expression—which in the present case equals 1— must lie near the midpoint of $[p_{1,0}, p_{1,1}]$ for D to exceed both L and Π. Therefore let $p_{1,0} = 0.5$ and $p_{1,1} = 1.5$. Also, let $Y = 3$. We then have $L = [2.5 + 1.5(3^{0.5})]/3$, or 1.699, and $\Pi = 3/[1.5 + 0.5(3^{-0.5})]$, or 1.677. Yet $D = \exp\{(1/3)\log 3 + 1/6\}$, or 1.704.

The economic price index is monotonic in u^r for $u^r \in [u_0, u_1]$, since it equals,

$$EPI(\mathbf{p}_0, \mathbf{p}_1, u^r) = \frac{p_{1,1}\gamma_1 + p_2\gamma_2 + u^r(p_{1,1})^{\beta_1}(p_2)^{(1-\beta_1)}}{p_{1,0}\gamma_1 + p_2\gamma_2 + u^r(p_{1,0})^{\beta_1}(p_2)^{(1-\beta_1)}} \tag{A-6}$$

Hence, there exists no $u^r \in [u_0, u_1]$ that makes $EPI(\mathbf{p}_0, \mathbf{p}_1, u^r)$ equal a D that exceeds both L and Π. In fact, given the assumptions, the u^r corresponding to D is 2.85, while $u_0 = 2.121$ and $u_1 = 0.408$.

Note that D would have been in between L and Π if the price had changed from 0.5 to 1 or from 1 to 1.5. For a price change from 0.5 to 1 we have $\Pi < D < L$, with index values of: L = 1.374; Π = 1.359; and D = 1.369. Letting the price change from 1 to 1.5 reverses the inequalities, with L = 1.242, Π = 1.246, and D = 1.244.

Example of a Divisia Index for a Linear Demand Curve: Assume that only good i changes price and that its demand function is: $q_i(p_i,Y) = \alpha p_i + \delta Y + \gamma$. Good i becomes a luxury at prices above $-\gamma/\alpha$. Hausman (1981) shows that the conditional expenditure function $\hat{e}(p_i, u)$ is:

$$\hat{e}(p_i, u) = \exp(\delta p_i)u - \frac{1}{\delta}\left[\alpha p_i + \alpha/\delta + \gamma\right] \tag{A-7}$$

If income is constant, $\log D$ is just A/Y, where A denotes the Marshallian surplus. Integrating $q_i(p_i,\cdot)$ shows that $A = (\alpha\,\bar{p}_i + \delta Y + \gamma)(p_{i1} - p_{i0})$, where $\bar{p}_i$ denotes the average of p_{i1} and p_{i0}, the initial and final values of p_i. Hence

$$D = \exp\{\delta(p_{i1} - p_{i0})\}\left[1 + \frac{\alpha p_{i0} + \alpha/\delta + \gamma}{Y\delta}\right] - \frac{\alpha p_{i1} + \alpha/\delta + \gamma}{Y\delta}.$$

Define u_0 as the solution to $\hat{e}(p_{i0}, u_0) = Y$. Then evaluating $\hat{e}(p_{i1}, u_0)/\hat{e}(p_{i0}, u_0)$ gives a Laspeyres-Konüs index of:

$$L = \exp\{\delta(p_{i1} - p_{i0})\}\left[1 + \frac{\alpha p_{i0} + \alpha/\delta + \gamma}{Y\delta}\right] - \frac{\alpha p_{i1} + \alpha/\delta + \gamma}{Y\delta} \tag{A-8}$$

Similarly, evaluating $\hat{e}(p_{i1}, u_1)/\hat{e}(p_{i0}, u_1)$ gives a Paasche-Konüs index of:

$$\Pi = \frac{Y}{[Y + (1/d)(a\pi_{\iota 1} + a/d + g)]\varepsilon\xi\pi\{-d(\pi_{\iota 1}\pi_{\iota 0})\}(1/d)(a\pi_{\iota 0} + a/d + g)} \tag{A-9}$$

Hausman illustrates the calculation of the Hicksian and Marshallian surplus measures for a change in the price of gasoline from 75 cents to \$1.50 per gallon on the assumption that Y = \$720, α = -14.22, δ = 0.082 and γ = 4.95. Since both the initial and final price exceed the price of 34.8 cents at which gasoline becomes a luxury, the Divisia index for Hausman's illustration lies in between L and Π, with values of Π = 1.0510, D = 1.0513 and L = 1.0516. However a price change from 20 cents to 50 cents implies a Divisia index of 1.02489, compared with a value of 1.02490 for L and Π. Solving $EPI(0.2, 0.5, u^r)$ = 1.02489 implies that u^r = -1560, while u_0 = -1346.9 for the 20 cent price and u_1 = -1364.1 for the 50 cent price.

Derivation of Equation (14)*:* For PIGLOG demands, equation (12) has the form:

$$w(\mathbf{p},Y) = (\log Y)\,a(\mathbf{p}) + b(\mathbf{p}). \tag{A-10}$$

Muellbauer's (1975) Theorem 5 states that in the PIGLOG case, the expenditure function is of a form whose logarithm equals:

$$\log Y = u[\log H(\mathbf{p})] + \log B(\mathbf{p}). \tag{A-11}$$

Consequently, $\mathbf{w}_t \cdot \dot{\mathbf{p}}_t = \{u_t[\log H(\mathbf{p})] + \log B(\mathbf{p})\}\{a(\mathbf{p})\cdot\dot{\mathbf{p}}_t\} + b(\mathbf{p})\cdot\dot{\mathbf{p}}_t$. The numerator of equation (12) is then $(\mathbf{w}_t - \mathbf{w}_{0t})\cdot\dot{\mathbf{p}}_t = (u_t - u_0)[\log H(\mathbf{p})][a(\mathbf{p})\cdot\dot{\mathbf{p}}_t]$. Similarly, the denominator is $(\mathbf{w}_{1t} - \mathbf{w}_{0t})\cdot\dot{\mathbf{p}}_t = (u_t - u_0)[\log H(\mathbf{p})][a(\mathbf{p})\cdot\dot{\mathbf{p}}_t]$. Canceling like terms gives equation (13).

Figure 1: Paths Yielding Paasche-Konus and Laspeyres-Konus Indexes

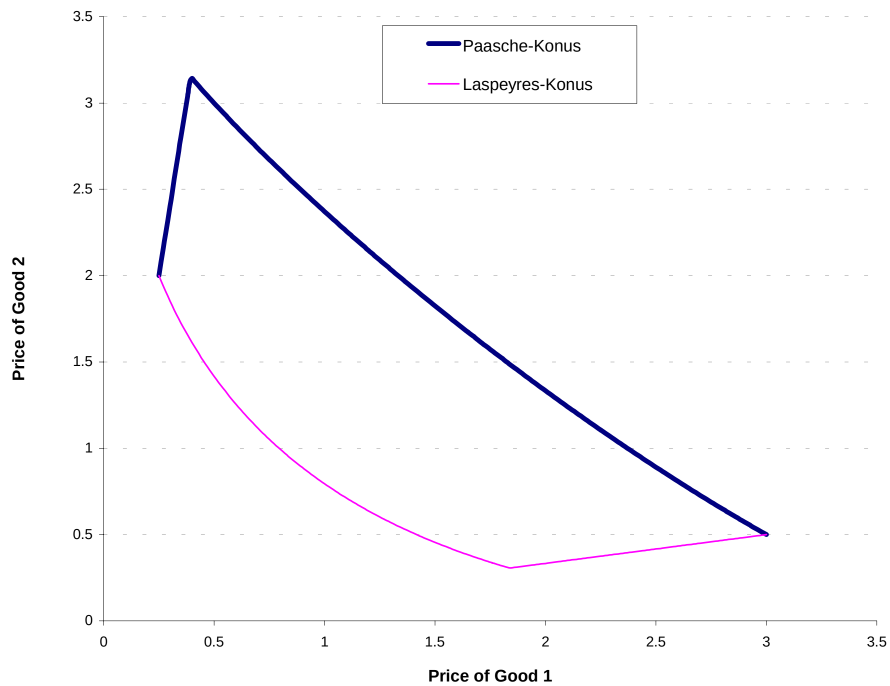

Figure 2: Ordinary and Compensated Integrands for First Example