

THE CONSUMER PRICE INDEX AND INDEX NUMBER PURPOSE

W. Erwin Diewert,
 Department of Economics,
 University of British Columbia,
 Vancouver, B. C.,
 Canada, V6T 1Z1.
 Email: diewert@econ.ubc.ca

Paper presented at the Fifth Meeting of the International Working Group on Price Indices (The Ottawa Group), Reykjavik, Iceland, August 25-27, 1999; Revised: December, 1999.

1. INTRODUCTION¹

“What index numbers are ‘best’? Naturally much depends on the purpose in view.” Irving Fisher (1921; 533).

A Consumer Price Index (CPI) is used for a multiplicity of purposes. Some of the more important uses are:

- as a compensation index; i.e., as an escalator for payments of various kinds;
- as a Cost of Living Index (COLI); i.e., as a measure of the relative cost of achieving the same standard of living (or utility level in the terminology of economics) when a consumer (or group of consumers) faces two different sets of prices;
- as a consumption deflator; i.e., it is the price change component of the decomposition of a ratio of consumption expenditures pertaining to two periods into price and quantity change components;
- as a measure of general inflation.

The CPI’s constructed to serve purposes (ii) and (iii) above are generally based on the *economic approach* to index number theory. Examples of CPI’s constructed to serve purpose (iv) above are the *harmonized indexes* produced for the member states of the European Union and the new *harmonized index of consumer prices for the euro area*, which pertains to the 11 members of the European Union that will use a common currency (called Euroland in the popular press).

The general purpose of this paper is to look at CPI construction from the viewpoint of index number purpose; i.e., given that a CPI is to be constructed for any one of the four purposes listed above, what index number formula seems “best” for this purpose.

The first purpose listed above is a bit too broad for us to consider in the present paper; however, Triplett (1983) very ably surveys this purpose.² Thus we restrict our attention to the remaining three purposes.

¹ The author thanks the SSHRC of Canada for financial support and all members of the Ottawa Group for their helpful comments on an earlier version of this paper. Particular thanks are due to John Astin, Bert Balk, Jörgen Dalén, Denis Fixler, John Greenlees, Johannes Hoffman, Rósmundur Guðnason, Robin Lowe, Carl Obst, Mick Silver, Jack Triplett, Keith Woolford and Kim Zieschang.

² See also Diewert and Fox (1998) for an exposition of the related theory of the income deflator.

In section 2 below, we consider an approach to implementing the Cost of Living concept. Our suggested approach leads to the Fisher (1922) Ideal price index.

Sections 3 and 4 look at the CPI from the viewpoint of value deflation. Section 3 takes a consumer theory approach while section 4 takes a producer theory approach.

Sections 2 to 4 are all based on economic approaches to index number theory; i.e., the theoretical index number that the CPI is supposed to approximate in these approaches is based on the assumption of optimizing behavior on the part of consumers or producers. Section 5 presents a brief survey of other approaches to index number theory that do not rely on the assumption of maximizing behavior. It is shown in section 5 that these alternative approaches all lead to the same class of index number formulae.

Section 6 looks at the CPI as a measure of inflation. Recent papers by Astin (1999) and Berglund (1999) are very useful in laying out *the theory of the harmonized index of consumer prices* (HICP) for Euroland. The recent papers of Woolford (1999) and Hill (1999a) are also useful for describing the properties of a *harmonized price index* or a “*pure*” *measure of price change*. We focus on the main differences between a harmonized price index and a Cost of Living Index in this section.

During the discussion that followed the presentation of this paper at the Ottawa Group meeting in Iceland, it became apparent that many price statisticians were very uncomfortable with the economic approach to index number theory, due perhaps to the overly formalistic presentation of the theory or the “unrealistic” nature of the assumptions made.³ These skeptical price statisticians were much more comfortable with the *fixed basket approach* to index number theory that is generally favored by proponents of the harmonized index approach to the measurement of price change. The fixed basket approach was termed a *pure price index* by many members of the Ottawa Group. In section 7, we will look at the *theory of the pure price index* from the viewpoint of the test or axiomatic approach to index number theory.

One of the main differences between a harmonized index and an economic index is that harmonized indexes are generally based on a *money outlays* or *money purchases* or *cost of acquisition* concept, while indexes based on producer or consumer theory are based on the *service flows* that can be attributed to the purchase of a consumer durable. In section 8, we examine more closely this difference between the harmonized and economic approaches.

Section 9 lists of some of the limitations of the economic approach to index number theory and section 10 concludes.

2. THE CPI AS A COST OF LIVING INDEX

“Simply put, it is: in constructing an index number to measure changes in the cost of living, and assuming only a single index number is to be prepared, whose cost of living should one have in mind?”

³ See Turvey (1999) in particular.

It is generally accepted in practice that some average of a selected group in the population is to be considered, but little attention has been given to the precise method of calculating this average." S. J. Prais (1959; 126).

In this section, we will consider an economic approach to the CPI that is based on the *plutocratic cost of living index* that was originally defined by Prais (1959). This concept was further refined by Pollak (1980; 276) (1981; 328) who defined his *Scitovsky-Laspeyres cost of living index* as the ratio of total expenditure required to enable each household in the economy under consideration to attain its base period indifference surface at period 1 prices to that required at period 0 prices. Diewert (1983a; 190-192) generalized Pollak's analysis and we further generalize Diewert's approach below.

Suppose there are N commodities in the economy in periods 0 and 1 that households consume *and* that we wish to include in our definition of the cost of living.⁴ Denote an N dimensional vector of commodity consumption in a given period by $q \equiv (q_1, q_2, \dots, q_N)$. Denote the vector of period t market prices facing each household by $p^t \equiv (p_1^t, p_2^t, \dots, p_N^t)$ for $t = 0, 1$. In addition to the market commodities that are in the vector q , we assume that each household is affected by an M dimensional vector of *environmental*⁵ or *demographic*⁶ variables or *public goods*, $e \equiv (e_1, e_2, \dots, e_M)$. We suppose that there are H households in the economy during periods 0 and 1 and the preferences of household h over different combinations of market commodities q and environmental variables e can be represented by the continuous utility function $f^h(q, e)$ for $h = 1, 2, \dots, H$.⁷ For periods $t = 0, 1$ and for households $h = 1, 2, \dots, H$, it is assumed that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t, \dots, q_{hN}^t)$ is a solution to the following household h expenditure minimization problem:

$$(1) \min_q \{ p^t \bullet q : f^h(q, e_h^t) \geq u_h^t \} \equiv C^h(u_h^t, e_h^t, p^t) ; t = 0, 1; h = 1, 2, \dots, H$$

where e_h^t is the environmental vector facing household h in period t , $u_h^t \equiv f^h(q_h^t, e_h^t)$ is the utility level achieved by household h during period t and C^h is the cost or expenditure function that is dual to the utility function f^h .⁸ Basically, these assumptions mean that each household has stable preferences over the same list of commodities during the two periods under consideration, the same households appear in each period and each household chooses its consumption bundle in the most cost efficient way during each period, conditional on the environmental vector that it faces during each period. Also, it is assumed that each household faces the same vector of prices during each period.

⁴ Choosing the domain of definition for a cost of living index is a nontrivial matter. We discuss this further in section 6 below.

⁵ This is the terminology used by Pollak (1989; 181) in his model of the conditional cost of living concept.

⁶ Caves, Christensen and Diewert (1982; 1409) used the terms *demographic variables* or *public goods* to describe the vector of conditioning variables e in their generalized model of the Konüs price index or cost of living index.

⁷ We assume that each $f^h(q, u)$ is continuous and increasing in the components of q and u and is quasiconcave in the components of q . See Diewert (1993a; 440) for background information on the meaning of these assumptions.

⁸ For background material on duality theory and its application to index number theory, see Diewert (1978b) (1993a). Note that $p \bullet q \equiv \sum_{n=1}^N p_n q_n$ is the inner product between the vectors p and q .

With the above assumptions in mind, we generalize Pollak (1980) (1981) and Diewert (1983a; 1990)⁹ and define the class of *conditional plutocratic cost of living indexes*, $P^*(p^0, p^1, u, e_1, e_2, \dots, e_H)$, pertaining to periods 0 and 1 for the arbitrary utility vector of household utilities $u \equiv (u_1, u_2, \dots, u_H)$ and for the arbitrary vectors of household environmental variables e_h for $h = 1, 2, \dots, H$ as follows:

$$(2) P^*(p^0, p^1, u, e_1, e_2, \dots, e_H) \equiv \sum_{h=1}^H C^h(u_h, e_h, p^1) / \sum_{h=1}^H C^h(u_h, e_h, p^0).$$

The numerator on the right hand side of (2) is the sum over households of the minimum cost, $C^h(u_h, e_h, p^1)$, for household h to achieve the arbitrary utility level u_h , given that the household h faces the arbitrary vector of household h environmental variables e_h and also faces the period 1 vector of prices p^1 . The denominator on the right hand side of (2) is the sum over households of the minimum cost, $C^h(u_h, e_h, p^0)$, for household h to achieve the same arbitrary utility level u_h , given that the household faces the same arbitrary vector of household h environmental variables e_h and also faces the period 0 vector of prices p^0 .

We now specialize the general definition (2) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^0, u_2^0, \dots, u_H^0)$ or the period 1 vector of household utilities $u^1 \equiv (u_1^1, u_2^1, \dots, u_H^1)$. We also specialize the general definition (2) by replacing the general household environmental vectors $(e_1, e_2, \dots, e_H) \equiv e$ by either the period 0 vector of household environmental variables $e^0 \equiv (e_1^0, e_2^0, \dots, e_H^0)$ or the period 1 vector of household environmental variables $e^1 \equiv (e_1^1, e_2^1, \dots, e_H^1)$. The choice of the base period vector of utility levels and base period environmental variables leads to the *Laspeyres conditional plutocratic cost of living index*, $P^*(p^0, p^1, u^0, e^0)$ ¹⁰, while the choice of the period 1 vector of utility levels and period 1 environmental variables leads to the *Paasche conditional plutocratic cost of living index*, $P^*(p^0, p^1, u^1, e^1)$. It turns out that these last two indexes satisfy some interesting inequalities.

Before we establish these inequalities, we require a few more definitions. Define the *aggregate period 0 and period 1 consumption vectors*, q^0 and q^1 , in the obvious way by summing over households in each period:

$$(3) q^0 \equiv \sum_{h=1}^H q_h^0 ; q^1 \equiv \sum_{h=1}^H q_h^1.$$

Once the aggregate consumption vectors for periods 0 and 1 have been defined by (3), we can define the *aggregate Laspeyres and Paasche price indexes*, P_L and P_P , as follows:

$$(4) P_L \equiv p^1 \bullet q^0 / p^0 \bullet q^0 ;$$

⁹ These authors provided generalisations of the plutocratic cost of living index due to Prais (1959). Pollak and Diewert did not include the environmental variables in their definitions of a group cost of living index.

¹⁰ This is the concept of a cost of living index that Triplett (1999; 27) finds most useful for measuring inflation: "One might want to produce a COL *conditional* on the base period's weather experience.... In this case, the unusually cold winter does not affect the *conditional* COL subindex that holds the environment constant. ... the COL subindex that holds the environment constant is probably the COL concept that is most useful for an anti-inflation policy." Hill (1999a; 4) endorses this point of view.

$$(5) P_P \equiv p^1 \bullet q^1 / p^0 \bullet q^1 .$$

Using definition (2), the *Laspeyres plutocratic conditional cost of living index*, $P^*(p^0, p^1, u^0, e^0)$, may be written as follows:

$$\begin{aligned} P^*(p^0, p^1, u^0, e^0) &\equiv \sum_{h=1}^H C^h(u_h^0, e_h^0, p^1) / \sum_{h=1}^H C^h(u_h^0, e_h^0, p^0) \\ &= \sum_{h=1}^H C^h(u_h^0, e_h^0, p^1) / \sum_{h=1}^H p^0 \bullet q_h^0 \quad \text{using (1) for } t = 0 \\ &\leq \sum_{h=1}^H p^1 \bullet q_h^0 / \sum_{h=1}^H p^0 \bullet q_h^0 \\ &\quad \text{since } C^h(u_h^0, e_h^0, p^1) \equiv \min_q \{ p^1 \bullet q : f^h(q, e_h^0) \geq u_h^0 \} \leq p^1 \bullet q_h^0 \text{ and } q_h^0 \\ &\quad \text{is feasible for the cost minimization problem for } h = 1, 2, \dots, H \\ &= p^1 \bullet \sum_{h=1}^H q_h^0 / p^0 \bullet \sum_{h=1}^H q_h^0 \\ &= p^1 \bullet q^0 / p^0 \bullet q^0 \quad \text{using (3)} \\ (6) \quad &= P_L \quad \text{using (4).} \end{aligned}$$

Thus the theoretical Laspeyres plutocratic conditional cost of living index, $P^*(p^0, p^1, u^0, e^0)$, is bounded from above by the observable aggregate Laspeyres price index P_L .¹¹ The inequality (6) was first obtained by Pollak (1989; 182) for the case of one household with environmental variables and by Pollak (1980; 276)¹² for the many household case but where the environmental variables are absent from the household utility and cost functions.

In a similar manner, using definition (2), the *Paasche conditional plutocratic cost of living index*, $P^*(p^0, p^1, u^1, e^1)$, may be written as follows:

$$\begin{aligned} P^*(p^0, p^1, u^1, e^1) &\equiv \sum_{h=1}^H C^h(u_h^1, e_h^1, p^1) / \sum_{h=1}^H C^h(u_h^1, e_h^1, p^0) \\ &= \sum_{h=1}^H p^1 \bullet q_h^1 / \sum_{h=1}^H C^h(u_h^1, e_h^1, p^0) \quad \text{using (1) for } t = 1 \\ &\geq \sum_{h=1}^H p^1 \bullet q_h^1 / \sum_{h=1}^H p^0 \bullet q_h^1 \\ &\quad \text{since } C^h(u_h^1, e_h^1, p^0) \equiv \min_q \{ p^0 \bullet q : f^h(q, e_h^1) \geq u_h^1 \} \leq p^0 \bullet q_h^1 \text{ and } q_h^1 \text{ is} \\ &\quad \text{feasible for the cost minimization problem for } h = 1, 2, \dots, H; \text{ hence} \\ &\quad 1 / C^h(u_h^1, e_h^1, p^0) \geq 1 / p^0 \bullet q_h^1 \text{ for } h = 1, 2, \dots, H. \\ &= p^1 \bullet \sum_{h=1}^H q_h^1 / p^0 \bullet \sum_{h=1}^H q_h^1 \end{aligned}$$

¹¹ Ralph Turvey notes that the usual aggregate Laspeyres index that is estimated by statistical agencies does *not* have the form P_L defined by (6). The problem is that the weight reference period does not usually coincide with the base price reference period for statistical agency price indexes. The resulting hybrid Laspeyres index has not been systematically studied by index number theorists, possibly because it does not have very transparent properties.

¹² This is Pollak's inequality between his Scitovsky-Laspeyres social cost of living index and his social Laspeyres index, which is our P_L .

$$\begin{aligned}
&= p^1 \bullet q^1 / p^0 \bullet q^1 && \text{using (3)} \\
(7) \quad &= P_P && \text{using (5)}.
\end{aligned}$$

Thus the theoretical Paasche conditional plutocratic cost of living index, $P^*(p^0, p^1, u^1, e^1)$, is bounded from below by the observable aggregate Paasche price index P_P . Diewert (1983a; 191) first obtained the inequality (7) for the case where the environmental variables are absent from the household utility and cost functions.

Using the inequalities (6) and (7) and the continuity properties of the conditional plutocratic cost of living $P^*(p^0, p^1, u, e)$ defined by (2), it is possible to modify the method of proof used by Konüs (1924) and Diewert (1983a; 191) and establish the following result:

Proposition 1: Under our assumptions, there exists a reference utility vector $u^* \equiv (u_1^*, u_2^*, \dots, u_H^*)$ such that the household h reference utility level u_h^* lies between the household h period 0 and 1 utility levels, u_h^0 and u_h^1 respectively for $h = 1, \dots, H$, and there exist household environmental vectors $e_h^* \equiv (e_{h1}^*, e_{h2}^*, \dots, e_{hM}^*)$ such that the household h reference m th environmental variable e_{hm}^* lies between the household h period 0 and 1 levels for the m th environmental variable, e_{hm}^0 and e_{hm}^1 respectively for $m = 1, 2, \dots, M$ and $h = 1, \dots, H$, and the conditional plutocratic cost of living index $P^*(p^0, p^1, u^*, e^*)$ evaluated at this intermediate reference utility vector u^* and the intermediate reference vector of household environmental variables $e^* \equiv (e_1^*, e_2^*, \dots, e_H^*)$ lies between the observable (in principle) aggregate Laspeyres and Paasche price indexes, P_L and P_P , defined above by (4) and (5).

Note that if market prices are identical for the two periods being compared, then $P_L = P_P = 1$, so the theoretical conditional plutocratic index $P^*(p^0, p^1, u^*, e^*)$ described in the above Proposition must also equal 1. Similarly, if prices are proportional for the two periods so that $p^1 = \lambda p^0$, then P_L and P_P both equal the factor of proportionality λ and the theoretical index $P^*(p^0, p^1, u^*, e^*)$ must also equal λ .

In the general case, the above Proposition says that a theoretical economic cost of living index for a group of households lies between the observable Paasche and Laspeyres price indexes that make use of the aggregate price and quantity vectors that pertain to that group of households. If we want a point estimate for this theoretical index, a reasonable strategy is to take a symmetric average of P_L and P_P as a point estimate. Examples of such symmetric averages¹³ are the arithmetic mean, which leads to the Sidgwick (1883; 68) Bowley (1901; 227)¹⁴ index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher¹⁵ (1922) ideal index, P_F defined as

$$(8) \quad P_F(p^0, p^1, q^0, q^1) \equiv [P_L(p^0, p^1, q^0, q^1) P_P(p^0, p^1, q^0, q^1)]^{1/2}.$$

What is the “best” symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under

¹³ For a discussion of the properties of symmetric averages, see Diewert (1993b).

¹⁴ See Diewert (1993c; 36) for additional references to the early history of index number theory.

¹⁵ Bowley (1899; 641) appears to have been the first to suggest the use of this index.

consideration to satisfy the *time reversal test*¹⁶. We say that the index number formula $P(p^0, p^1, q^0, q^1)$ satisfies this test if

$$(9) P(p^1, p^0, q^1, q^0) = 1 / P(p^0, p^1, q^0, q^1) ;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1, p^0, q^1, q^0)$ is equal to the reciprocal of the original index $P(p^0, p^1, q^0, q^1)$.

Diewert (1997; 138) proved the following result:

Proposition 2: The Fisher Ideal price index defined by (8) above is the *only* index that is a symmetric average of the Laspeyres and Paasche price indexes, P_L and P_P , and satisfies the time reversal test (9) above.

Thus the economic approach to the cost of living index that we have outlined in this section leads to the Fisher ideal index as the “best” functional form.¹⁷

What is the *purpose* of the index described in this section? *The purpose of the plutocratic cost of living index is to provide a single summary measure of the amount of price change over a well defined domain of definition of commodities that a well defined group of households has experienced over two periods of time. Utility levels and environmental variables are held constant at intermediate reference levels; only the vector of market prices varies between the base and comparison periods.*

We turn now to theories of the CPI as a deflator for consumption expenditures.

3. THE CPI AS A DEFLATOR: A CONSUMER THEORY APPROACH

“As we have seen, the cost of living index provides a precise answer to a narrow and specific question. If one wishes to compare expenditures required to attain a particular base indifference curve at two sets of prices, then, by definition, the cost of living index is the appropriate index. But price indexes are often used to deflate an index of total expenditure to obtain an index of quantity or ‘real consumption’. ...we examine the conditions under which the preference field quantity index coincides with the quantity index obtained by using the cost of living index to deflate an index of expenditure.” Robert A. Pollak (1983; 133).

Under the assumptions made in the previous section, aggregate household expenditures in period 0 are $p^0 \bullet \sum_{h=1}^H q_h^0 = p^0 \bullet q^0$ and in period 1 are $p^1 \bullet \sum_{h=1}^H q_h^1 = p^1 \bullet q^1$. The *deflation problem* is the problem of choosing a *price index* $P(p^1, p^0, q^1, q^0)$ and a *quantity index* $Q(p^1, p^0, q^1, q^0)$ such that the observed expenditure ratio for

¹⁶ See Diewert (1992; 218) for early references to this test. If we want our price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes in which case, satisfaction of the time reversal test is not so important.

¹⁷ Caves, Christensen and Diewert (1982; 1409-1411) developed an alternative approach to the conditional COL (but only for the single household case) based on the assumption of translog preferences, which led to the Törnqvist price index P_T defined later in section 5 below.

periods 0 and 1, $p^1 \bullet q^1 / p^0 \bullet q^0$, is equal to the product of the price and quantity indexes; i.e., we want to find suitable functions P and Q such that

$$(10) \quad p^1 \bullet q^1 / p^0 \bullet q^0 = P(p^0, p^1, q^0, q^1) Q(p^0, p^1, q^0, q^1).$$

Since the left hand side of (10) is in principle observable, it can be seen that if we determine *either* the functional form for the price index $P(p^0, p^1, q^0, q^1)$ *or* the functional form for the quantity index $Q(p^0, p^1, q^0, q^1)$, then the functional form for the remaining function is automatically determined. Thus the deflation problem boils down to choosing *one* of the two functions, $P(p^0, p^1, q^0, q^1)$ or $Q(p^0, p^1, q^0, q^1)$.

Obviously, we could pick the conditional plutocratic cost of living index, $P^*(p^0, p^1, u^*, e^*)$, discussed in the previous section as a theoretically appropriate price deflator function. As indicated in the previous section, an approximation to this theoretical index is the Fisher ideal price index $P_F(p^0, p^1, q^0, q^1)$ defined by (8) above. Thus a theoretical cost of living index and practical approximations to it can serve as price deflator functions.

It is also possible to pick an appropriate theoretical quantity index (and practical approximations to it) as the quantity deflator $Q(p^0, p^1, q^0, q^1)$ and then the price index $P(p^0, p^1, q^0, q^1)$ defined residually using (10) could be defined to be the price deflator function. We will follow the strategy of picking a quantity index in the present section and let the price deflator function be determined residually.

We make the same assumptions about our H households as in the previous section, *except we drop the environmental variables from the household utility functions*.¹⁸ Thus, we define the family of *generalized Allen (1949) quantity indexes*, $Q^*(u^0, u^1, p)$, for the reference vector of commodity prices $p \equiv (p_1, p_2, \dots, p_N)$ as follows:

$$(11) \quad Q^*(u^0, u^1, p) \equiv \sum_{h=1}^H C^h(u_h^1, p) / \sum_{h=1}^H C^h(u_h^0, p)$$

where $u^0 \equiv (u_1^0, u_2^0, \dots, u_H^0)$ is the base period vector of household utilities and the period 1 vector of household utilities is $u^1 \equiv (u_1^1, u_2^1, \dots, u_H^1)$.¹⁹ Note that $u_h^t \equiv f^h(q_h^t)$ is the actual utility level attained by household h in period t.

It is instructive to compare the definition of the theoretical family of price indexes $P^*(p^0, p^1, u, e)$ defined by (2) above with the family of theoretical quantity indexes $Q^*(u^0, u^1, p)$ defined by (11): in (11), prices are held fixed at the reference price vector p while the utility quantities vary, while in (2), quantities are held fixed at the reference utility vector u while the prices vary. In (2), the environmental variables e_h

¹⁸ We could define a family of conditional generalized Allen quantity indexes, $Q^*(u^0, u^1, p, e)$, for the reference vector of prices p and the reference vector of environmental variables e as $Q^*(u^0, u^1, p, e) \equiv \sum_{h=1}^H C^h(u_h^1, e_h, p) / \sum_{h=1}^H C^h(u_h^0, e_h, p)$ where $u^0 \equiv (u_1^0, u_2^0, \dots, u_H^0)$ is the base period vector of household utilities and the period 1 vector of household utilities is $u^1 \equiv (u_1^1, u_2^1, \dots, u_H^1)$. However, specializing (p,e) to (p^0, e^0) does not lead to the usual Laspeyres bound (14) and specializing (p,e) to (p^1, e^1) does not lead to the usual Paasche bound (15).

¹⁹ If H=1, then (11) reduces to the definition of the Allen (1949) family of quantity indexes. If the division sign on the right hand side of (11) is replaced by a minus sign, then the resulting index reduces to a sum of Hicks' (1941-2; 128) *equivalent variations* if $p=p^0$ and to a sum of Hicks' (1941-2; 128) *compensating variations* if $p=p^1$. For the case of one household, Diewert (1981) compared Allen quantity indexes with Malmquist (1953) and implicit Konüs (1924) quantity indexes.

facing each household h are held constant over the two periods being compared while in (11), there are no environmental variables.

Definition (11) involves a cardinalization of utility for each household. At the utility level u_h for household h , the cardinal measure of its utility is proportional to the size of the budget set that is tangent to the indifference surface indexed by u_h using the reference prices p to form the budget set. Samuelson (1974) referred to this cardinalization of utility as *money metric utility*.

Before we specialize the general definition of the consumer theory quantity index $Q^*(u^0, u^1, p)$ defined by (11) for the special cases where the vector of reference prices p equals the base period prices p^0 or the current period prices p^1 , we define the aggregate *Laspeyres and Paasche quantity indexes*, Q_L and Q_P , as follows:

$$(12) \quad Q_L \equiv p^0 \bullet q^1 / p^0 \bullet q^0 ;$$

$$(13) \quad Q_P \equiv p^1 \bullet q^1 / p^1 \bullet q^0 .$$

Specializing definition (11), the *Laspeyres Generalized Allen quantity index*, $Q^*(u^0, u^1, p^0)$, may be defined as follows:

$$\begin{aligned} Q^*(u^0, u^1, p^0) &\equiv \sum_{h=1}^H C^h(u_h^1, p^0) / \sum_{h=1}^H C^h(u_h^0, p^0) \\ &= \sum_{h=1}^H C^h(u_h^1, p^0) / \sum_{h=1}^H p^0 \bullet q_h^0 \quad \text{using (1) for } t = 0 \\ &\leq \sum_{h=1}^H p^0 \bullet q_h^1 / \sum_{h=1}^H p^0 \bullet q_h^0 \\ &\quad \text{since } C^h(u_h^1, p^0) \equiv \min_q \{ p^0 \bullet q : f^h(q) \geq u_h^1 \} \leq p^0 \bullet q_h^1 \text{ and } q_h^1 \text{ is} \\ &\quad \text{feasible for the cost minimization problem for } h = 1, 2, \dots, H \\ &= p^0 \bullet \sum_{h=1}^H q_h^1 / p^0 \bullet \sum_{h=1}^H q_h^0 \\ &= p^0 \bullet q^1 / p^0 \bullet q^0 \quad \text{using (3)} \\ (14) \quad &= Q_L \quad \text{using (13).} \end{aligned}$$

Thus the theoretical Laspeyres Generalized Allen quantity index, $Q^*(u^0, u^1, p^0)$, is bounded from above by the observable aggregate Laspeyres quantity index Q_L .

In a similar manner, by specializing definition (11), the *Paasche Generalized Allen quantity index*, $Q^*(u^0, u^1, p^1)$, may be defined as follows:

$$\begin{aligned} Q^*(u^0, u^1, p^1) &\equiv \sum_{h=1}^H C^h(u_h^1, p^1) / \sum_{h=1}^H C^h(u_h^0, p^1) \\ &= \sum_{h=1}^H p^1 \bullet q_h^1 / \sum_{h=1}^H C^h(u_h^0, p^1) \quad \text{using (1) for } t = 1 \\ &\geq \sum_{h=1}^H p^1 \bullet q_h^1 / \sum_{h=1}^H p^1 \bullet q_h^0 \\ &\quad \text{since } C^h(u_h^0, p^1) \equiv \min_q \{ p^1 \bullet q : f^h(q) \geq u_h^0 \} \leq p^1 \bullet q_h^0 \text{ and } q_h^0 \text{ is} \end{aligned}$$

feasible for the cost minimization problem for $h = 1, 2, \dots, H$; hence
 $1/C^h(u_h^0, p^1) \geq 1/p^1 \bullet q_h^0$ for $h = 1, 2, \dots, H$.

$$\begin{aligned}
 &= p^1 \bullet \sum_{h=1}^H q_h^1 / p^1 \bullet \sum_{h=1}^H q_h^0 \\
 &= p^1 \bullet q^1 / p^1 \bullet q^0 && \text{using (3)} \\
 (15) \quad &= Q_P && \text{using (13)}.
 \end{aligned}$$

Thus the theoretical Paasche Generalized Allen quantity index, $Q^*(u^0, u^1, p^1)$, is bounded from below by the observable (in principle) aggregate Paasche quantity index Q_P .

Using the inequalities (14) and (15) and the continuity properties of the generalized Allen quantity index $Q^*(u^0, u^1, p)$ defined by (11), it is possible to adapt a proof used by Diewert (1983a; 218)²⁰ and establish the following result:

Proposition 3: Under our assumptions, there exists a reference price vector $p^* \equiv (p_1^*, p_2^*, \dots, p_N^*)$ such that the reference price for commodity n , p_n^* , lies between the price of commodity n in periods 0 and 1, p_n^0 and p_n^1 respectively for $n = 1, \dots, N$, and the generalized Allen quantity index $Q^*(u^0, u^1, p^*)$ evaluated at this reference commodity price vector p^* lies between the observable (in principle) aggregate Laspeyres and Paasche quantity indexes, Q_L and Q_P , defined above by (12) and (13).

The above Proposition says that a theoretical economic quantity index for a group of households lies between the observable Paasche and Laspeyres quantity indexes that make use of the aggregate price and quantity vectors that pertain to that group of households. If we want a point estimate for this theoretical index, a reasonable strategy is to take a symmetric average of Q_L and Q_P as a point estimate. The Fisher ideal quantity index, Q_F , is defined as the geometric mean of the Laspeyres and Paasche quantity indexes, Q_L and Q_P :

$$(16) \quad Q_F(p^0, p^1, q^0, q^1) \equiv [Q_L(p^0, p^1, q^0, q^1) Q_P(p^0, p^1, q^0, q^1)]^{1/2}.$$

As in the previous section, it is very desirable for a quantity index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*. We say that the quantity index number formula $Q(p^0, p^1, q^0, q^1)$ satisfies this test if

$$(17) \quad Q(p^1, p^0, q^1, q^0) = 1/Q(p^0, p^1, q^0, q^1);$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $Q(p^1, p^0, q^1, q^0)$ is equal to the reciprocal of the original index $Q(p^0, p^1, q^0, q^1)$.

²⁰ Define $h(\lambda) \equiv Q^*(u^0, u^1, (1-\lambda)p^0 + \lambda p^1)$ and adapt the rest of the proof of Theorem 14 in Diewert (1983a; 218-219). Alternatively, define $g(\lambda) \equiv Q^*(u^0, u^1, (1-\lambda)p^1 + \lambda p^0)$ and adapt the proof of Proposition 1 in the Appendix below.

It is straightforward to use the proof in Diewert (1997; 138) and prove the following result:

Proposition 4: The Fisher Ideal quantity index defined by (16) above is the *only* index that is a symmetric average of the Laspeyres and Paasche quantity indexes, Q_L and Q_P , and satisfies the time reversal test (17) above.

Thus a practical “best” approximation to a theoretical quantity index of the type defined by (11) above is the Fisher ideal quantity index Q_F defined by (16). This result is the quantity index analogue to the price index result that we obtained in the previous section.

What are the implications of Proposition 4 for the functional form for the price deflator? Using the adding up identity (10)²¹, it can be seen that the price index $P(p^0, p^1, q^0, q^1)$ that corresponds to the Fisher Quantity index Q_F is

$$(18) \quad P(p^0, p^1, q^0, q^1) = [p^1 \bullet q^1 / p^0 \bullet q^0] / Q_F(p^0, p^1, q^0, q^1) \\ = P_F(p^0, p^1, q^0, q^1)$$

using definitions (8) and (16). Thus the price index that is implicitly defined by the adding up identity (10) and the Fisher ideal quantity index turns out to be the Fisher ideal price index, a well known result.²²

Thus both of the economic approaches to the price index considered in sections 2 and 3 have led us to the Fisher ideal price index P_F as a good approximation to the underlying theoretical indexes.

What is the *purpose* of the price index described in this section? *The purpose of the price index $P(p^0, p^1, q^0, q^1)$ which occurs in the value change equation (10) above is to act as a deflator which converts the nominal change in expenditures by a well defined group of households over a well defined set of commodities over two periods into a real change in expenditures.* In this section, we have defined the price index residually, and defined the quantity index to be a generalized Allen quantity index of the type defined by (11).

The approaches to the CPI considered in sections 2 and 3 have relied on the assumption of utility maximizing (and cost minimizing) behavior on the part of households. In the next section, we consider the problem of deflating consumer expenditures from the viewpoint of producer theory.

4. THE CPI AS A DEFLATOR: A PRODUCER THEORY APPROACH

“We assume that the firm is a price taker and that the base period and the comparison period output price vectors are p^0 and p^1 respectively. Natural choices for the output price index numbers are then $P_0^0(p^1, p^0, x^0)$ and $P_0^1(p^1, p^0, x^1)$. The first compares the revenue obtainable by using x^0 [the base period

²¹ This is known as the weak factor reversal test or the product test in the index number literature.

²² See Fisher (1911; 398) (1922; 142).

input vector]with the base period technology when the price vectors are p^1 and p^0 , and the second compares the revenue obtainable by using x^1 [the comparison period input vector] with the comparison period technology when the price vectors are p^1 and p^0 . Thus the first uses the Laspeyres perspective, and the second uses the Paasche perspective. If we have no preference for either we can opt for a symmetric mean of both index numbers.” Bert M. Balk (1998; 85).

In this section, we use the theory of the producer price index to construct a theoretical consumption price deflator.

We suppose that there are F firms in the market sector of the economy that are producing the N commodities that are demanded by households (and possibly other sectors of the economy). We suppose that for $f = 1, 2, \dots, F$, the feasible set of outputs and inputs for firm f in period t is a set S_f^t , a subset of $N + M$ dimensional space. Thus if (q, x) belongs to the set S_f^t , then the vector of (net) consumption outputs $q = (q_1, \dots, q_N)$ can be produced by firm f in period t if the vector of (net) inputs $x = (x_1, \dots, x_M)$ is available for use.²³ We make the following conventions on the quantities q_{fn} of net output n for firm f : if commodity n for firm f is an output in period t , then q_{fn}^t is positive and if it is an input in period t , then q_{fn}^t is negative. Similarly, we make the following conventions on the quantities x_{fm} of net input m for firm f : if commodity m for firm f is an input in period t , then x_{fm}^t is positive and if it is an output in period t , then x_{fm}^t is negative. With these conventions, the sum of price times quantity over all commodities in the set of consumption outputs in period t for firm f , is $\sum_n p_{fn}^t q_{fn}^t \equiv p_f^t \bullet q_f^t$.

Let $p = (p_1, \dots, p_N)$ denote a positive vector of consumption output prices that producers in the market sector might face in period t .²⁴ Then firm f 's *consumption revenue function* using its period t technology and the net input vector x is defined as:

$$(19) \pi_f^t(p, x) \equiv \max_q \{ p \bullet q : (q, x) \text{ belongs to } S_f^t \}; f = 1, 2, \dots, F$$

where as usual $p \bullet q = \sum_n p_n q_n$ denotes the inner product of the vectors p and q . Thus $\pi_f^t(p, x)$ is the maximum value of consumption output, $\sum_n p_n q_n$, that firm f can produce, given that the vector of net inputs x is available for use, using its period t technology.²⁵

Denote the vector of period t final demand consumption prices facing each firm by $p^t \equiv (p_1^t, p_2^t, \dots, p_N^t)$ for $t = 0, 1$. In this section, we assume that the observed firm f production vector for finally demanded consumption commodities q_f^t is a solution to the following firm f revenue maximization problem:

$$(20) \max_q \{ p^t \bullet q : (q, x_f^t) \text{ belongs to } S_f^t \} = p^t \bullet q_f^t = \pi_f^t(p^t, x_f^t); f = 1, 2, \dots, F; t = 0, 1;$$

where x_f^t is the observed period t net input vector for firm f .

²³ Some of the components of the x vector could be environmental variables. Note that nonconsumption outputs produced by firms are included as (negative) components of the x vectors.

²⁴ The consumption price vector in this section will generally be different from the consumption price vectors that appeared in the previous two sections by the amounts of commodity taxes that create differences between the prices that consumers face versus the corresponding prices that producers face.

²⁵ For the case of one producer, the function π^t is known as the GDP function or the national product function in the international trade literature; see Kohli (1978)(1991) or Woodland (1982).

It will be useful to aggregate over the F firms and define the *aggregate market sector period 0 and period 1 consumption vectors*, q^0 and q^1 , in the obvious way by summing over firms in each period:

$$(21) \quad q^0 \equiv \sum_{f=1}^F q_f^0 ; q^1 \equiv \sum_{f=1}^F q_f^1.$$

With the above preliminary definitions and assumptions, it is now possible to use the revenue functions π_f^t to define the economy's period t technology *market sector consumption price index* P^t between periods 0 and 1 as follows:

$$(22) \quad P^t(p^0, p^1, x) = \sum_{f=1}^F \pi_f^t(p^1, x_f) / \sum_{f=1}^F \pi_f^t(p^0, x_f)$$

where p^t is the vector of consumption output prices that the market sector faces in period t , $t = 0, 1$, and $x \equiv (x_1, x_2, \dots, x_F)$ is a reference vector of net inputs for the F firms in the market sector.²⁶ If $N = 1$ so that there is only one consumption output in the economy, then it can be shown that the consumption output price index collapses down to the single consumption price ratio between periods 0 and 1, p_1^1 / p_1^0 .

Note that there are a wide variety of price indexes of the form (22) depending on which (t, x) reference technology and reference net input vector x that we choose. Usually, we are interested in two special cases of the general definition of the consumption output price index (22): (a) $P^0(p^0, p^1, x^0)$ which uses the period 0 technology set and the net input vector $x^0 \equiv (x_1^0, x_2^0, \dots, x_F^0)$ that was actually used in period 0 and (b) $P^1(p^0, p^1, x^1)$ which uses the period 1 technology set and the net input vector $x^1 \equiv (x_1^1, x_2^1, \dots, x_F^1)$ that was actually used in period 1.

Recall that q^0 and q^1 are the observed consumption output vectors for the market sector in periods 0 and 1 respectively defined by (21) above. Under our consumption revenue maximising assumptions (20), we can show that the two theoretical indexes, $P^0(p^0, p^1, x^0)$ and $P^1(p^0, p^1, x^1)$ described in (a) and (b) above, satisfy the following inequalities (23) and (24)²⁷:

$$\begin{aligned} (23) \quad P^0(p^0, p^1, x^0) &\equiv \sum_{f=1}^F \pi_f^0(p^1, x_f^0) / \sum_{f=1}^F \pi_f^0(p^0, x_f^0) && \text{using definition (22)} \\ &= \sum_{f=1}^F \pi_f^0(p^1, x_f^0) / \sum_{f=1}^F p^0 \bullet q_f^0 && \text{using (20)} \\ &= \sum_{f=1}^F \pi_f^0(p^1, x_f^0) / p^0 \bullet q^0 && \text{using (21)} \\ &\geq \sum_{f=1}^F p^1 \bullet q_f^0 / p^0 \bullet q^0 \\ &\quad \text{since } q_f^0 \text{ is feasible for the maximisation problem} \\ &\quad \text{which defines } \pi_f^0(p^1, x_f^0) \text{ and so } \pi_f^0(p^1, x_f^0) \geq p^1 \bullet q_f^0 \\ &= p^1 \bullet q^0 / p^0 \bullet q^0 && \text{using (21)} \\ &\equiv P_L \end{aligned}$$

where P_L is the aggregate market sector Laspeyres producer price index for consumption commodities. Similarly, we have:

²⁶ For the case of one firm, this concept of the consumption output price index (or a closely related variant) was defined by Fisher and Shell (1972), Samuelson and Swamy (1974; 588-592), Archibald (1977; 60-61), Diewert (1980; 461)(1983b; 1055) and Balk (1998; 83-89).

²⁷ Fisher and Shell (1972; 57-58) and Archibald (1977; 66) established these inequalities for the case of a single firm.

$$\begin{aligned}
(24) \quad P^1(p^0, p^1, x^1) &\equiv \sum_{f=1}^F \pi_f^1(p^1, x_f^1) / \sum_{f=1}^F \pi_f^1(p^0, x_f^1) && \text{using definition (22)} \\
&= \sum_{f=1}^F p^1 \bullet q_f^1 / \sum_{f=1}^F \pi_f^1(p^0, x_f^1) && \text{using (20)} \\
&= p^1 \bullet q^1 / \sum_{f=1}^F \pi_f^1(p^0, x_f^1) && \text{using (21)} \\
&\leq p^1 \bullet q^1 / \sum_{f=1}^F p^0 \bullet q_f^1 \\
&\quad \text{since } q_f^1 \text{ is feasible for the maximisation problem} \\
&\quad \text{which defines } \pi_f^1(p^0, x_f^1) \text{ and so } \pi_f^1(p^0, x_f^1) \geq p^0 \bullet q_f^1 \\
&= p^1 \bullet q^1 / p^0 \bullet q^1 && \text{using (21)} \\
&\equiv P_P
\end{aligned}$$

where P_P is the aggregate market sector Paasche producer price index for the consumption component of final demand. Thus the inequality (23) says that the observable Laspeyres index of consumption output prices P_L is a *lower bound* to the theoretical consumption output price index $P^0(p^0, p^1, x^0)$ and (24) says that the observable Paasche index of consumption output prices P_P is an *upper bound* to the theoretical consumption output price index $P^1(p^0, p^1, x^1)$. Note that these inequalities are in the opposite direction compared to their counterparts in the theory of the true cost of living index outlined in section 2 above.²⁸

It is possible to define a theoretical producer output price index that falls *between* the observable Paasche and Laspeyres price indexes. To do this, first we define a hypothetical aggregate consumption revenue function, $\pi(p, \alpha)$, that corresponds to the use of an α weighted average of the firm f technology sets S_f^0 and S_f^1 for periods 0 and 1 as the reference technology sets and that uses an α weighted average of the period 0 and period 1 firm f net input vectors x_f^0 and x_f^1 as the reference input vectors²⁹:

$$\begin{aligned}
(25) \quad \pi(p, \alpha) \\
&\equiv \max_{q^s} \{ \sum_{f=1}^F p \bullet q_f : (q_f, \{1-\alpha\}x_f^0 + \alpha x_f^1) \text{ belongs to } (1-\alpha)S_f^0 + \alpha S_f^1; f=1,2,\dots,F \}.
\end{aligned}$$

Thus the consumption revenue maximisation problem in (25) corresponds to the use by firm f of an average of its period 0 and 1 input vectors x_f^0 and x_f^1 where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α and firm f uses an ‘‘average’’ of the period 0 and period 1 technology sets, S_f^0 and S_f^1 respectively, where the period 0 set gets the weight $1 - \alpha$ and the period 1 set gets the weight α , and α is a number between 0 and 1.³⁰ We can now use the new hypothetical consumption revenue function defined by (25) in order to define the following family (indexed by α) of theoretical net output price indexes:

$$(26) \quad P(p^0, p^1, \alpha) \equiv \pi(p^1, \alpha) / \pi(p^0, \alpha).$$

The important advantage that theoretical consumption output price indexes of the form defined by (22) or (26) have over the traditional Laspeyres and Paasche output price indexes P_L and P_P is that the theoretical indexes deal adequately with

²⁸ This is due to the fact that the optimisation problem in the cost of living theory is a cost *minimization* problem as opposed to our present net revenue *maximization* problem.

²⁹ The function $\pi(p, \alpha)$ also depends on the period 0 and period 1 net input vectors x_f^0 and x_f^1 for periods 0 and 1 for each firm f and on the firm production possibilities sets S_f^0 and S_f^1 for each period.

³⁰ When $\alpha=0$, $\pi(p, 0) = \sum_{f=1}^F \pi_f^0(p, x_f^0)$ and when $\alpha = 1$, $\pi(p, 1) = \sum_{f=1}^F \pi_f^1(p, x_f^1)$.

substitution effects; i.e., when an output price increases, the producer supply should increase, holding inputs and the technology constant.³¹

It is possible to modify a proof in Diewert (1983b; 1060-1061) and show that the following result is true:

Proposition 5: There exists an α between 0 and 1 such that the theoretical consumption output price index defined by (26) lies between the observable (in principle) Paasche and Laspeyres output price indexes, P_P and P_L ; i.e., there exists an α such that

$$(27) \quad P_L \equiv p^1 \bullet q^0 / p^0 \bullet q^0 \leq P(p^0, p^1, \alpha) \leq p^1 \bullet q^1 / p^0 \bullet q^1 \equiv P_P \quad \text{or} \quad P_P \leq P(p^0, p^1, \alpha) \leq P_L$$

The fact that the Paasche and Laspeyres output price indexes provide upper and lower bounds to a “true” output price $P(p^0, p^1, \alpha)$ in (27) is a more useful and important result than the one sided bounds on the “true” indexes that were derived in (23) and (24) above; if the Paasche and Laspeyres indexes are numerically close to each other, then we know that a “true” economic price index is fairly well determined and we can find a reasonably close approximation to the “true” index by taking a symmetric average of P_L and P_P . As in Propositions 2 and 4 above, it can be argued that the “best” symmetric average of P_L and P_P to take is the geometric average, which again leads to Irving Fisher’s (1922) ideal price index, P_F :

$$(28) \quad P_F(p^0, p^1, q^0, q^1) \equiv [P_L(p^0, p^1, q^0, q^1) P_P(p^0, p^1, q^0, q^1)]^{1/2}.$$

Thus, usually, the Fisher ideal index P_F will be a reasonably good approximation to an economic price index based on producer theory, the degree of approximation becoming closer as the producer price Paasche and Laspeyres indexes, P_P and P_L , are closer to each other.

Thus the economic approach to the CPI that is based on producer theory also leads to a Fisher ideal price index as being a good approximation to the corresponding theoretical price index. However, note that the prices, which appear in this section, are *producer prices*; i.e., they do not include any tax wedges that fall between producers and consumers. Also, the domains of definition for the consumer oriented price indexes in the previous two sections and the producer oriented price index in the present section are *different* in general. In the consumer case, we are summing *demands* over the H households in our domain of definition while in the producer case, we are summing *supplies* of the N consumption commodities over the F firms or production units in our domain of definition.

³¹ This is a normal output substitution effect. However, in the real world, one can often observe period to period increases in price that are *not* accompanied with a corresponding increase in supply. We rationalise these abnormal substitution effects by hypothesising that they are caused by technological progress. For example, suppose the price of computer chips decreases substantially going from period 0 to 1. If the technology were constant over these two periods, we would expect production of home computers to decrease going from period 0 to 1. In actual fact, the opposite happens but what has happened is that technological progress has led to a sharp reduction in the cost of producing home computers, which is passed on to the final demanders of computers.

What is the *purpose* of the index described in this section? *The purpose of the output price index defined in this section is to provide a single summary measure of the amount of price change over a well defined domain of definition of commodities that a well defined group of production units has experienced over two periods of time. In making the price comparison, only the price vectors are allowed to change over the two periods; the technology sets and input vectors of the firms are held constant at some intermediate technology sets and input vectors. The output price index can also act as a deflator which converts the nominal change in revenues received by a well defined group of firms or production units over a well defined set of commodities over two periods into a real change in revenues.* With respect to the second purpose described above, recall equation (10) above. This equation can be applied in the present context except that the aggregate (over households) expenditure ratio, $p^1 \bullet q^1 / p^0 \bullet q^0$, is now interpreted as an aggregate (over firms) revenue ratio. *Thus the purpose of the price index $P(p^0, p^1, q^0, q^1)$ which occurs in the value change equation (10) above and is approximated by P_F defined by (28) is to act as a deflator which converts the nominal change in revenues by a well defined group of production units over a well defined set of commodities over two periods into a real change in revenues.*

Many price statisticians find the economic approach to the determination of an appropriate functional form for the consumer price index to be overly formalistic and intuitively implausible.³² Thus in the following section, we consider some alternative approaches for determining the functional form for the CPI that are perhaps more acceptable to these skeptical price statisticians.

5. OTHER APPROACHES TO INDEX NUMBER THEORY

“There are two fundamentally different ways in which the problem of price index numbers may be approached. We term them the *atomistic* [or stochastic and test approaches] and the *functional* [or economic] approaches. ... First we have what Edgeworth called the *indefinite standard* approach, which may more appropriately be called the *stochastic* approach. Here the assumption is made that any change that takes place in the ‘price level’ *ought*, so to speak, to manifest itself as a proportional change of all prices. ... According to this conception, the deviation of the individual price changes from proportionality must be considered more or less as errors of observation. But then the application of the theory of errors should enable us to determine the underlying proportionality factor. ... Another attempt to escape indeterminateness—while still employing the atomistic viewpoint—is the *test approach*. It consists in formulating certain formal tests regarding the function that expresses the price level change from one situation to another. ... In the *functional approach*, prices and quantities are looked upon as connected by certain—in point of principle, *observable*—relations. Here we do not—as in the stochastic approach—make the assumption that ideally the individual prices ought to change in the *same* proportion as we pass from one situation to another. We face the deviations from proportionality and take them merely as expressions for those *systematic* relations that serve to give an economic meaning to the index number.” Ragnar Frisch (1936; 3-10).

³² Fisher also thought that the economic approach based on utility maximization was useless as the following quotation indicates: “Since we cannot measure utility statistically, we cannot measure the ‘benefits of progress’. In the absence of statistical measurement, any practicable correction is out of the question. The ‘utility standard’ is therefore impracticable, even if the theory of such a standard were tenable.” Irving Fisher (1911; 222). Of course, modern developments in consumer theory and in the economic approach to index number theory do make it possible to measure utility to some degree of approximation.

In this section, we consider three alternatives to the economic approach to the determination of the consumer price index.³³ These three alternatives are:

- the fixed basket approach;
- the test approach and
- the stochastic approach.

Approach 1: The Fixed Basket Approach

This first alternative approach to measuring aggregate consumer price change between periods 0 and 1 dates back several hundred years³⁴. The fixed basket approach sets the CPI equal to the ratio of the costs of buying the same basket of goods in period 1 to period 0. There are two natural choices for the reference basket: the period 0 commodity vector q^0 or the period 1 commodity vector q^1 . These two choices lead to the Laspeyres price index P_L defined earlier by (4) and the Paasche price index P_P defined by (5). The problem with these index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. As was noted in Proposition 2 above, if we want our price index to satisfy the time reversal test, then we are led to the Fisher ideal price index P_F defined by (8) above as the ‘best’ estimator of price change from the viewpoint of the symmetrically weighted fixed basket approach to index number theory.

It is interesting to note that this symmetric basket approach to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

“If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean ... as a first approximation.” A. L. Bowley (1901; 227).

“When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. ... They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found.” A. L. Bowley (1919; 348).³⁵

In section 7 below, we will study fixed basket indexes from a slightly different perspective.

Approach 2: The Test or Axiomatic Approach

If there is only one commodity, then a very reasonable measure of price change going from period 0 to 1 is just the relative price of the single commodity, p_1^1 / p_1^0 . Note that this functional form for the price index when $N = 1$ satisfies the time reversal test,

³³ Additional alternative approaches are reviewed in Diewert (1981; 201-208) (1993c; 42-43).

³⁴ See Diewert (1993c; 34-36) (1997; 128-129) for more information on this approach.

³⁵ Fisher (1911; 417-418) (1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

(9) above. Note also that p_1^1/p_1^0 is increasing and homogeneous of degree one in p_1^1 and is decreasing and homogeneous of degree minus one in p_1^0 . Now let the number of commodities N be greater than 1. The test approach asks that the price index $P(p^0, p^1, q^0, q^1)$ satisfy mathematical properties that are analogues to the properties of the single commodity price index. For example, we can ask that $P(p^0, p^1, q^0, q^1)$ satisfy the time reversal test (9) or that $P(p^0, \lambda p^1, q^0, q^1) = \lambda P(p^0, p^1, q^0, q^1)$ where λ is a positive number or that $P(p^0, p^1, q^0, q^1)$ assume that P satisfies enough “reasonable” tests or properties so that the functional form for P is determined.

There is not complete agreement on just what are the “reasonable” tests that an index number formula P should satisfy. However, the current consensus seems to be that the Fisher ideal price index P_F satisfies more “reasonable” axioms than its competitors³⁶.

Thus the test approach leads to the Fisher ideal index as being the “best” functional form.

Approach 3: The Stochastic or Statistical Approach

The stochastic approach to the determination of the price index can be traced back to the work of Jevons and Edgeworth over a hundred years ago³⁷.

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for $n = 1, 2, \dots, N$ can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., it is assumed that

$$(29) \quad p_n^1/p_n^0 = \alpha + \varepsilon_n \quad ; \quad n = 1, 2, \dots, N$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2 . The least squares or maximum likelihood estimator for α is the Carli price index P_C defined as

$$(30) \quad P_C(p^0, p^1) \equiv \sum_{n=1}^N (1/N) p_n^1/p_n^0.$$

Unfortunately, P_C does not satisfy the time reversal test, $P_C(p^1, p^0) = 1/P_C(p^0, p^1)$ ³⁸.

Let us change our stochastic specification as follows: assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

$$(31) \quad \ln(p_n^1/p_n^0) = \beta + \varepsilon_n \quad ; \quad n = 1, 2, \dots, N$$

where $\beta \equiv \ln \alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2 . The least squares or maximum likelihood estimator for β is the

³⁶ Diewert (1992) showed that the Fisher ideal price index satisfies 20 “reasonable” tests, which is more than its competitors satisfy. For a comprehensive review of the test approach, see Balk (1995).

³⁷ For references to the literature, see Diewert (1993c; 37-38) (1995a).

³⁸ In fact Fisher (1922; 66) noted that $P_C(p^0, p^1) P_C(p^1, p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0 ; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula.

logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α^{39} is the Jevons price index P_J defined as:

$$(32) P_J(p^0, p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}.$$

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C . However, both the Jevons and Carli price indexes suffer from a fatal flaw: each price relative p_n^1/p_n^0 is regarded as being equally important and is given an equal weight in the index number formulae (30) and (32). Keynes was particularly critical of this *unweighted stochastic approach* to index number theory. He directed the following criticism towards this approach, which was vigorously advocated by Edgeworth (1923):

“Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The ‘errors of observation’, the ‘faulty shots aimed at a single bull’s eye’ conception of the index number of prices, Edgeworth’s ‘objective mean variation of general prices’, is the result of confusion of thought. There is no bull’s eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.

What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the ‘mean’ are ‘random’ in the sense required by the theory of the combination of independent observations. In this theory the divergence of one ‘observation’ from the true position is assumed to have no influence on the divergences of other ‘observations’. But in the case of prices, a movement in the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of ‘independence’, there is between the ‘errors’ in the successive ‘observations’ what some writers on probability have called ‘connexity’, or, as Lexis expressed it, there is ‘sub-normal dispersion’.

We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity.” John Maynard Keynes (1930; 76-77).

The main point Keynes seemed to be making in the above quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, we can interpret Keynes as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; i.e., through the workings of the general equilibrium system. Thus Keynes seemed to be leaning towards the economic approach to index number theory (even before it was even developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in the above quotation is that

³⁹ Greenlees (1998) pointed out that although $(1/N) \sum_{n=1}^N \ln(p_n^1/p_n^0)$ is an unbiased estimator for β , the corresponding exponential of this estimator, P_J defined by (32), will generally *not* be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_n = \ln p_n^1/p_n^0$. Taking expectations, we have: $E x_n = \beta = \ln \alpha$. Define the positive, convex function f of one variable x by $f(x) \equiv e^x$. By Jensen’s (1906) inequality, we have $E f(x) \leq f(E x)$. Letting x equal the random variable x_n , this inequality becomes: $E(p_n^1/p_n^0) = E f(x_n) \leq f(E x_n) = f(\beta) = e^\beta = e^{\ln \alpha} = \alpha$. Thus for each n , we have $E(p_n^1/p_n^0) \leq \alpha$, and it can be seen that the Jevons price index defined by (32) will generally have a downward bias.

there is no such thing as the inflation rate; there are only price changes that pertain to well specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified.⁴⁰ A final point that Keynes made is that price movements must be weighted by their economic importance; i.e., by quantities or expenditures.

In addition to the above theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth's unweighted stochastic approach:

“The Jevons—Edgeworth ‘objective mean variation of general prices’, or ‘indefinite’ standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.

Finally, the conclusion that all the standards ‘come to much the same thing in the end’ has been reinforced ‘inductively’ by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions. ... On the contrary, the tables give above (pp. 53,55) supply strong presumptive evidence that over long period as well as over short period the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent.” John Maynard Keynes (1930; 80-81).

In the above quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indexes of wholesale prices showed broadly similar movements. However, Keynes showed empirically that these wholesale price indexes moved quite differently than his consumer price indexes.⁴¹

In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.

Theil (1967; 136-137) proposed a solution to the lack of weighting in (32). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the n th price relative is equal to $s_n^0 \equiv p_n^0 q_n^0 / p^0 \bullet q^0$, the period 0 expenditure share for commodity n . Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0 \ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the

⁴⁰ We will return to this point in section 6 below.

⁴¹ Using the OECD national accounts data for the last four decades, some broad trends in the rates of increase in prices for the various components of GDP can be observed: rates of increase for the prices of internationally traded goods have been the lowest, followed by the prices of reproducible capital goods, followed by consumer prices, followed by wage rates. From other sources, land prices have shown the highest rate of price increase over this period. Of course, if a country adjusts the price of computer related equipment for quality improvements, then the aggregate price of capital machinery and equipment tends to move *downwards* in recent years. Thus there are long term systematic differences in price movements over different domains of definition.

overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1 \ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so as usual, we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change⁴². Theil (1967; 137) argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the n th price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n . Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

$$(33) \ln P_T(p^0, p^1, q^0, q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln(p_n^1/p_n^0).$$

Taking antilogs of both sides of (33), we obtain the Törnqvist (1936) (1937) Theil price index, P_T . This index number formula appears to be “best” from the viewpoint of Theil's stochastic approach to index number theory.⁴³

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Diewert (1995a) and Wynne (1997).

We can summarise the results of our review of the three alternative approaches to the determination of the index number formula for the CPI as follows: all three approaches lead to the choice of either the Fisher ideal formula P_F defined by (8) or the Törnqvist-Theil formula P_T defined in (33) as being “best”. From an empirical point of view, it will not matter very much whether P_F or P_T is chosen since the two indexes approximate each other to the second order around an equal price and quantity point⁴⁴ and thus the two indexes will generally approximate each other quite closely.

We turn now to the theory underlying “inflation” indexes or “harmonized” indexes.

6. THE CPI AS A MEASURE OF INFLATION

“Our next question is: What prices should be selected in constructing an index number? The answer to this question largely depends on the *purpose* of the index number. Hitherto we have considered only one purpose of an index number, viz. to best meet the requirements of the equation of exchange. But index numbers may be used for many other purposes, of which the two chief are to measure *capital* and to measure *income*. Each of the three purposes mentioned (viz. exchange, capital and income) may be subclassified according as the comparison desired is between *places* or *times*.” Irving Fisher (1911; 204-205).

⁴² This time, the “best” symmetric mean is the arithmetic mean since this choice leads to an index number formula that satisfies the time reversal test.

⁴³ This approach is probably not the last word in the specification of an adequate theoretical framework for the stochastic approach to index numbers but at least it deals with the objections of Keynes to the unweighted approach (provided that we have a well specified domain of definition for the stochastic index).

⁴⁴ See Diewert (1978a; 888). This technique for comparing index number formulae using first or second order Taylor series expansions of the formulae was worked out by Edgeworth (1901; 410-411) almost a century ago but was forgotten until the 1970's.

Central bankers are concerned with the measurement of inflation. But what is “inflation”? It is some sort of broad or general measure of price change occurring between two periods. But what exactly is the domain of definition of an “inflation” index; i.e., over what set of economic agents or institutional units and over what set of commodities and transactions will the index be defined?

Refer back again to equation (10), which provided a decomposition of a value ratio:

$$(34) p^1 \bullet q^1 / p^0 \bullet q^0 = P(p^0, p^1, q^0, q^1) Q(p^0, p^1, q^0, q^1).$$

Early “inflation” theories of the price index specified that the set of transactions that the value sums for periods 1 and 0, $p^1 \bullet q^1$ and $p^0 \bullet q^0$ respectively, should encompass *the set of all monetary transactions that occurred in the economy in periods 1 and 0*. This domain of definition for an “inflation” index dates back to Fisher⁴⁵ at least:

“Without attempting to construct index numbers which particular persons and classes might sometimes wish to take as standard, we shall merely inquire regarding the formation of such a general index number. It must, as has been pointed out, include all goods and services. But in what proportion shall these be weighted? How shall we decide how much weight should be given, in forming the index, to the *stock* of durable capital and how much weight to the *flow* of goods and services through a period of time, –the flow to individuals, which mirrors consumption? The two things are incommensurable. Shall we count the railways of the country as equally important with a month’s consumption of sugar, or with a year’s?”

To cut these Gordian knots, perhaps the best and most practical scheme is that which has been used in the explanation of the P in our equation of exchange [MV = PT], an index number in which every article and service is weighted according to the value of it *exchanged* at base prices in the year whose level of prices it is desired to find. By this means, goods bought for immediate consumption are included in the weighting, as are also all durable capital goods exchanged during the period covered by the index number. What is repaid in contracts so measured is the same *general purchasing power*. This includes purchasing power over everything purchased and purchasable, including real estate, securities, labor, other services, such as the services rendered by corporations, and commodities.” Irving Fisher (1911; 217-218).

However, under present economic conditions, this extremely broad definition of an “inflation” index has fallen out of favor due to the preponderance of transactions in currency and stock market trading, which totally overwhelm other more interesting transactions.⁴⁶ Thus it is necessary to narrow the scope of “all monetary transactions” to a smaller domain of definition that encompasses transactions over a specified set of

⁴⁵ Fisher’s (1911; 201) original choice of functional form for the price index in his equation of general purchasing power was the Paasche index.

⁴⁶ Fisher (1911; 225-226) noted that it would be difficult to obtain data for all transactions: “It is, of course, utterly impossible to secure data for all exchanges, nor would this be advisable. Only articles which are standardized, and only those the use of which remains through many years, are available and important enough to include. These specifications exclude real estate, and to some extent wages, retail prices, and securities, thus leaving practically nothing but wholesale prices of commodities to be included in the list of goods, the prices of which are to be compounded into an index number.” Fisher (1911; 226-227) went on to argue that for the United States in the early years of the century, real estate transactions amounted “only to a fraction of 1 per cent of the total transactions”, security transactions amounted to “about 8 per cent of the total transactions of the country”, wages “amount to about 3 per cent and retail prices could be omitted “because wholesale and retail prices roughly correspond in their movements”. Obviously, these rough approximations are no longer relevant. Note that Fisher wanted to exclude new commodities from his inflation index, a preference that is echoed by Hill (1999a; 6).

commodities and a specified set of transactors.⁴⁷ Choosing the set of transactions to be covered by the price index might be termed *the domain of definition problem*.⁴⁸

Due to the difficulties involved in defining a price index that is defined over all monetary transactions, we will restrict ourselves to a domain of definition that encompasses some subset of household purchases of consumer goods and services.⁴⁹ Recent papers by Astin (1999), Berglund (1999) and Woolford (1999) are helpful in providing some specific suggestions on what some of the characteristics of such consumer purchases inflation measure or harmonized index of consumer prices (HICP) should be. In their view, such an index should have the following properties:

- (a) It should encompass only market transactions⁵⁰; i.e., imputations such as user costs or rental prices for housing would not be included⁵¹.
- (b) It should not include interest rates⁵² or interest costs since “such costs are neither a good or a service but the instrument for balancing the supply and demand of money” [Berglund (1999; 3)].
- (c) The consumer purchases inflation index should include new purchases of dwelling units⁵³.
- (d) The harmonized index should use the Laspeyres formula but the basket must be updated between one and seven years with a preference for more frequent reweighting⁵⁴.
- (e) “Expenditure incurred for business purposes should be excluded.” [Berglund (1999; 6)].
- (f) The harmonized index of consumer prices (HICP) for a country should include the consumption expenditures made by foreign visitors and exclude the expenditure by residents while visiting in a foreign country⁵⁵.

⁴⁷ Note that in order to apply any of the approaches to index number theory that we have considered in this paper to the determination of the price index $P(p^0, p^1, q^0, q^1)$ that occurs in (34), it is necessary that : (a) the set of commodities be the same over the two periods under consideration and (b) that the transactions pertain to the same set of transactors. If these two conditions are not satisfied, then the meaning of the price index is not clear.

⁴⁸ As we have seen Fisher (1911; 204-230) provided an extensive discussion of the domain of definition problem as did Knibbs (1924; 47-49). Diewert (1997; 134-136) discussed alternative household consumption domains of definition. For example, he discussed whether seasonal goods should be excluded from the domain of definition, whether consumer durables should be excluded, whether future goods or savings should be included, whether leisure should be included, whether commodity taxes should be included, and whether commodities that have highly variable prices should be excluded.

⁴⁹ According to Berglund (1999; 3), the domain of definition for the European Union Harmonized CPI's is the “final monetary consumption expenditure of households.”

⁵⁰ See Woolford (1999; 1) and Astin (1999; 3).

⁵¹ “Firstly, the harmonized indices would be concerned only with actual monetary transactions. So, for example, in the area of housing costs, we would not use the imputed rents method to measure the ‘price’ of owner-occupied housing (such a method is motivated in the measurement of the volume of consumption of housing services, but is irrelevant in the context of measurement of price change).” Berglund (1999; 3).

⁵² See Woolford (1999; 1) and Astin (1999; 3).

⁵³ Woolford (1999; 19). At present, the harmonized price index excludes both dwelling services and purchases of new dwellings. “However, consideration is at present being given to a future possible inclusion of the net acquisition prices of new dwellings.” Berglund (1999; 5).

⁵⁴ See Astin (1999; 3-4) and Berglund (1999; 4). Berglund (1999; 4) notes that the Paasche formula is equally valid from a theoretical perspective but its use “is ruled out on practical grounds”.

⁵⁴ Berglund (1999; 6).

⁵⁵ Berglund (1999; 6).

(g) The prices, which should be used in an HICP, are consumer prices (or final demand prices) rather than producer prices.⁵⁶ Thus harmonized prices should include commodity and value added taxes in principle.

(h) A harmonized index should not include *new commodities* in the domain of definition of the price index; i.e., if a commodity is present in one of the two periods being compared but not the other, then that commodity should be excluded from the price index.⁵⁷

We discuss the last point first. In our presentation of the various economic approaches in sections 2 to 4 above, it was assumed that the list of commodities was the same in the two periods so that point (h) was not really addressed. However, it is safe to say that many believers in the economic approach to index number theory would agree that the Hicksian (1940; 114) reservation price technique is appropriate *in principle*, even though it may be difficult to implement in an objective and reproducible form. However, believers in the “inflation” index approach to index number theory have tended to restrict their index domains of definition to commodities that are present in both periods, as the following quotations indicate:

“It is, or course, utterly impossible to secure data for all exchanges, nor would this be advisable. Only articles which are standardized, and only those the use of which remains through many years, are available and important enough to include.” Irving Fisher (1911; 225).

“When price indexes are used in order to measure fluctuations in the purchasing efficiency of money, the prices should of course refer to identical commodities, that is, A on the first occasion must be identical with A’ on the second. In short the series of commodities must be the same at the compared dates, otherwise the results would be vitiated by confusions of kind, quality, etc.” Sir George H. Knibbs (1924; 48).

“From the point of view of users interested in inflation (i.e., price changes) the relevant COL may be defined to exclude the impact on welfare resulting from the introduction of a completely new good *on the grounds that the welfare benefit does not stem from a price reduction but from advances in knowledge and technology*.

This is a controversial topic. The conventional counter argument is that a price can be associated with a completely new good before it appears, namely the hypothetical demand reservation price—the lowest price that would reduce demand to zero. This must be higher than the price charged for a completely new good when it first appears (assuming some of the good is actually bought) so that a price reduction does occur. However, analysts and policy makers concerned about the general price level are not likely to be interested in purely hypothetical price reductions which do not actually occur, which cannot be estimated and which have no bearing on the demand for money.” Peter Hill (1999a; 7).

Hill’s points deserve some further discussion. His first point may be looked at from a different perspective. Suppose an advance in knowledge reduced the price of an existing standard product instead of leading to the production of a new commodity. Should we somehow exclude this lower price from the domain of definition of the

⁵⁶ According to Berglund (1999; 3), a harmonised CPI “shall be based on the price of goods and services available for purchase in the economic territory of the Member State for the purposes of directly satisfying consumer needs.”

⁵⁷ This point follows from point (a); i.e., that imputations should be avoided. Note however that this avoidance of imputations should apply to the problem of disappearing commodities as well as to the introduction of new commodities. This seemingly small point could have a large effect on the computation of harmonized indexes since a substantial fraction of price quotes disappear each year. Following the no imputations methodology, this would lead to a dramatic reduction in the number of price relatives that should appear in the harmonized index. In practice, the EU does allow for the replacement of old models by new models, so imputations are not entirely avoided.

COL? The answer most economists would give is no! Thus I do not find Hill's first point very convincing.

His second point is that central bankers fighting inflation are not likely to be interested in the including the effects of new commodities in the indexes that they watch. But is this really true? If the proportion of new commodities entering the marketplace each year is a small fraction of total transactions, then Hill's point is probably true. But what if the fraction of new commodities is significant and growing over time? In this situation, if the central bank attempts to stabilize a household consumer price index that excludes new goods, then there is some danger that this policy could in fact be *deflationary*.⁵⁸ It should be noted that the proportion of new commodities that are introduced to the marketplace each year does not have to be growing over time in order for a substantial new goods bias to occur; see Diewert (1987; 779). It seems very likely that the fraction of new commodities that are entering the marketplace each year is increasing:

"Is there any general evidence on the magnitude of the new products bias other than anecdotes? Some general evidence comes from two sources. The first source is the A. D. Nielsen scanner data base. William Hawkes has informed me that the number of U. S. Universal Product Codes has grown from 950,000 in January 1990 to 1,650,000 in September 1995. Some of this increase in products is simply a market penetration phenomenon: more and more manufacturers are coding their commodities. However, a substantial fraction of the above increase has to represent a genuine increase in consumers' choice sets. A second general source of evidence on the magnitude of the new products problem comes from the records of the BLS itself: each month, approximately 3 percent of the price quotes of the previous month simply disappear. A substantial fraction of these missing quotes is probably due to temporary inventory shortages and other factors, but surely a substantial fraction must be due to the replacement of old goods by newer goods." W. Erwin Diewert (1996a; 33).

The current business press is full of articles about the "new economy" where it is necessary for firms to develop new products and services and to compete globally. This focus on new products is a perhaps a natural outcome of the growth in the world marketplace, stimulated by reductions in transport costs and trade barriers; i.e., as the size of the market grows, it is inevitable that increased specialization will take place. There is also evidence from the automobile (and other) industries that the time to develop new models and get them on the marketplace has fallen dramatically in recent years. Thus it is becoming increasingly likely that traditional economic models that hold the list of commodities fixed over time are not relevant to today's economic conditions.⁵⁹ Here is the problem: if firms are collectively devoting an increasing fraction of resources to the development, production, distribution and marketing of new products but the statistical system *ignores* the immediate welfare improvements that these new products generate, how can the public regard the resulting harmonized price and volume measures as being accurate measures of economic reality?⁶⁰

⁵⁸ Perhaps the difference in views between Hill and Diewert could be resolved as follows: the statistical agency could compute the CPI a la Hill and omit new (and disappearing) commodities. However, a supplementary calculation could be made available to interested parties that attempted to make some adjustment for the new goods problem.

⁵⁹ The theoretical economics literature is starting to develop models that have new goods as their focus; e.g., see Romer (1994).

⁶⁰ It should be noted that the HICP does *not* ignore the existence of new products: member states of the EU are *required* to take active steps to ensure that "newly significant" goods and services are introduced into the index as soon as there is evidence that they have a market penetration of more than 0.1% of total HICP coverage. However, the Hicksian increments of consumer surplus as consumers go from zero units of a new commodity to one or more units are ignored.

We turn now to a discussion of properties (a) to (g) above for a harmonized price index. These 7 properties for a consumer purchases index or a harmonized index of consumer prices index (HICP) enable us to distinguish it from a cost of living (COL) index of the type discussed in section 2 above or from a producer price index (PPI) of the type discussed in section 4 above. A harmonized index shares properties (e) and (g) with a COLI and a HCPI also shares properties (a), (b) and (c) with a PPI.⁶¹ However properties (a)-(c) are not consistent with a COL index, which should use either a rental equivalence approach to the consumption of housing services or a user cost approach to the consumption of owner-occupied housing services. In section 8 below, we will *contrast* the money purchases approach to housing with the user cost approach.

There are some fairly severe problems associated with all three classes of index in determining the appropriate set of transactions that should be included in the transaction domains of definition for the indexes. High levels of income taxes and commodity taxes in many countries are driving consumers to engage in an increasing amount of household production. This household production could lead to the production of commodities, which are sold on the market (e.g., self employment income or various types of business services), and the corresponding revenues and the associated inputs should be excluded from both the COL and the HICP. On the other hand, this household production could simply lead to the production of various consumer goods and services (e.g., the household production of wine or beer or a home renovation) and in this case, the inputs used by the household should appear in the list of commodities that are finally demanded by households for consumption purposes. A similar difficult domain of definition problem occurs when forming a producer price index for sales to final demand: sales of “consumer commodities” to other producers should be excluded from the index. Finally, in calculating a COL for a group of households who are resident in a country, the foreign tourist purchases of these households should in principle appear in the list of commodities consumed. However, foreign purchases that are incurred while conducting business abroad should be excluded from the COL but included as imports in a complete system of producer price indexes. In practice, it is very difficult to make the above distinctions.

As noted in point (d) above, a harmonized price index is based on the use of a Laspeyres formula; i.e., the base period quantity basket is repriced over time. It is argued that this must be done on practical grounds. However, this argument is not completely convincing since Shapiro and Wilcox (1997) have shown that the Lloyd (1975) Moulton (1996) formula can be used to form a close approximation to a superlative index like the Fisher ideal or Törnqvist-Theil. The Lloyd-Moulton formula makes use of the same information set as the usual Laspeyres index except that an estimate of the elasticity of substitution between the various commodities must be provided to the statistical agency.⁶²

Since John Astin has been in charge of the development of the Harmonized Indices of Consumer Prices (HICP) for Eurostat, it may be useful to examine his recent paper in some detail, since this paper explains some of the reasons for the various choices that

⁶¹ Sales of new dwelling units would be included in a PPI.

⁶² John Astin points out that it may not be very easy to obtain reliable estimates of this elasticity of substitution.

had to be made in order to get HICP off the ground. Hence below, we quote Astin extensively on four topics and give our reactions to each topic.

- **Imputations and the treatment of interest.**

“In practice, ‘inflation’ is what happens to be the index used to measure it! We decided at an early stage that inflation is essentially a monetary phenomenon. It concerns the changing power of money to produce goods and services. This led us down two important paths. Firstly, the HICPs would be concerned only with actual monetary transactions. So, for example, in the field of housing, we would not use the imputed rents method to measure the price of owner-occupied housing. (This is a valuable concept in the context of the measurement of the volume of consumption of housing services, but it is irrelevant in the context of the measurement of price change). Secondly, we would not include the cost of borrowed money, which is neither a good nor a service. So interest payments were to be excluded. This immediately set the HICP apart from some national CPIs which include interest payments on the grounds that they form part of the regular outgoings of households: a perfectly reasonable argument in the context of a compensation index, but less so for an inflation index.” John Astin (1999; 2-3).

Thus the European Union HCIP can only have actual transactions that took place in the two periods being compared in its domain of definition and thus there are to be no imputed prices in the index. We have already stressed that the domain of definition problem needs to be very carefully specified. However, the above description of the HICP does not explain why monetary transactions in certain classes of consumption goods were *excluded* from the domain of definition of the HICP; i.e., why were *actual transactions* in second hand houses excluded? On the other hand, in a Cost of Living Index (COLI) approach to the Consumer Price Index (CPI), the consumption of owner-occupied housing would be valued according to a *rental equivalence approach* or a *user cost approach*. In the rental equivalence approach, the services of an owner-occupied home would be valued at a comparable market rental price. It is true that this price would be an *imputed* or *estimated* one but is this a very different procedure from say *estimating* the aggregate price of television sets in a country from say 30 representative price quotes? It is true that homes are a more complex product but it seems to me that the two estimation or imputation situations are not all that different.⁶³ On the other hand, in the user cost approach to the purchase of a consumer durable, it is explicitly recognized that *not all of the good is consumed in the period of purchase*. Thus the purchase price should be decomposed into two parts: the first part which is the cost to the consumer of using the *services of the commodity* during the period of purchase, and a second part, which is a form of *investment* that will yield either a return or services to the consumer in future periods. Moreover, the user cost approach provides us with a way of valuing the services of the older vintages of household consumer durable goods and thus allows us to build up a more comprehensive picture of actual household consumption as opposed to the money purchases approach advocated for the HICP, which includes only *new* purchases of consumer durables. In order to estimate these user costs, it is necessary to have information on the prices of used consumer durables at the beginning and end of each period. Thus one could argue that the user cost approach uses *more* information on actual asset transactions than the money purchases or acquisitions

⁶³ But Ralph Turvey responds as follows: “But there is a difference in principle between estimating what could be observed with more effort, TV prices, and what can never be observed, imputed rents.” John Astin responds to this argument as follows: “In the first case you are using a standard statistical sampling technique to estimate a real population attribute. In the second, you are first of all inventing a transaction where none exists, and then using a debatable method of imputing a ‘price’ to that nonexistent transaction. I think the two are hardly comparable.”

approach to the treatment of durables. We will return to a more technical discussion of these alternative approaches in section 7 below.

This is perhaps not the appropriate place to get into an extensive discussion of the role of interest in economics but many economists would be somewhat puzzled at the meaning of the statement that interest is the cost of borrowed money and hence is not a good or a service. Most economists would regard interest as the payment for the use of financial capital for a specified period of time and hence regard it as a service. During the discussion of the preliminary version of this paper, Keith Woolford brought out an interesting reason for the possible exclusion of interest from a price index. Namely, interest is not a *contemporaneous price*; i.e., an interest rate necessarily refers to *two* points in time; a beginning point when the capital is loaned and an ending point when the capital must be repaid. Thus if for some reason, one wanted to restrict attention to a domain of definition that consisted of only contemporaneous prices, interest rates would be excluded. However, in general, interest rates *are* prices (even though they are more complex than contemporaneous prices): interest is the payment for the use of the services of financial capital for a specified period of time. The interesting question is how *precisely* should interest be entered into a consumer price index? We provide an answer to this question in section 8 below when we discuss user costs.

Even if interest does play a role in the user cost treatment of consumer durables in a COLI approach to the CPI, it must be recognized that many central bankers desire statistical agencies to produce a consumer price index that *excludes* interest rates. In a personal communication, John Astin summarized the situation as follows:

“However, quite apart from the theoretical aspects, it is a fact that the inclusion of interest payments in a CPI leads to enormous difficulties from the point of view of those responsible for handling monetary policy. To give but one example, in the UK, where mortgage interest plays an important part in the housing sub-index, the vagaries of interest rates and their perverse effect on the overall index in regard to anti-inflation policy have led the government to abandon the RPI as a tool for monitoring inflation in the context of anti-inflation policy and to use instead RPI-X, which excludes mortgage interest. The European Central Bank, for precisely those reasons, also strongly supported our proposal to exclude interest payments from the HICP.”

- **The treatment of nonmarket or highly subsidized services.**

“In most cases goods and services on the market are sold at a price determined by normal market processes. But in several important sectors, especially healthcare and education, it is common to have partial or total subsidies provided by the state. This raises difficult problems in CPI construction, regarding both concept and measurement.

Some experts argued that the full, unsubsidised, price of such products should be included. ...

Others argued that the HICP does not aim to measure total inflation, but just that part impacting on the private household sector ...

The solution finally adopted owes much to the work of Peter Hill. He showed that within the ESA [European System of Accounts] structure it was possible to define an element of expenditure, which he named HFMCF, which related solely to that part of the expenditure actually paid by private households. So that, for example, if 80% of a chemist’s prescription charge is reimbursed by the government, only the remaining 20% would be included in the HICP. A change in the subsidy would have a similar effect on the ‘market’ price to a change in VAT [Value Added Tax], which, of course, is also included in all CPIs.” John Astin (1999; 5).

The treatment of subsidized goods chosen by the HICP is exactly the right one if our domain of definition is the transactions of *households*, (which is a consumer theory

perspective). However, if our domain of definition is the consumer goods and services produced by *firms*, then the treatment is not correct. From this perspective (a producer theory perspective), the “correct” price is the full, unsubsidized price. The European Union HICP does, in most instances⁶⁴, take the consumer theory perspective; i.e., its stated domain of definition is the (consumption) transactions of member nation households. Thus HICP treatment of subsidized goods seems to be quite correct from the consumer perspective.

- **The treatment of owner-occupied housing.**

“A special coverage problem concerns owner-occupied housing. This has always been one of the most difficult sectors to deal with in CPIs.

Strictly, the price of housing should not be included in a CPI because it is classified as capital. On the other hand, the national accounts classifies imputed rents of owner-occupiers as part of consumers’ expenditure. This is a reasonable thing to do if the aim is to measure the volume of consumption of the capital resource of housing. But that is not what a CPI is measuring. ...

So, after many hours of debate, the Working Party came to the conclusion that there were just two options. The first was to simply exclude owner-occupied housing from the HICP. One could at least argue that this was a form of harmonization, although it is worrying that there are such large differences between Member States in the percentages of the population which own or rent their dwellings. ...

The second option was to include owner-occupied housing on the basis of acquisition costs, essentially treating them like any other durable. Most secondhand housing would be excluded: in practice the index would include new houses plus a small volume of housing new to the household sector (sales from the company or government sectors to the household sector).

The main problem here is practical: several countries do not have new house price indices and their construction could be difficult and costly. A Task Force is at present examining these matters. Final recommendations are due at the end of 1999.” John Astin (1999; 6).

Excluding owner-occupied housing from a CPI would give a very incomplete picture of the price movements facing households in the country. Suppose a sudden asset bubble developed in the prices of houses (as seems to be happening in England right now). In the short run, rents would be contractually fixed and would not reflect the asset price bubble. Thus omitting owner-occupied housing from the CPI would give a very misleading picture of “inflation” facing consumers. On the other hand, taking an acquisitions cost or money purchases point of view to housing also has its problems. As Astin noted above, most of the purchase cost of a new house has the character of a capital investment; only a small part of the purchase price can be attributed to consumption of housing services in the current period. There is also the problem of neglecting the stock of used houses in this approach. Nevertheless, this second approach seems preferable to the first approach, which just omits owner-occupied housing from the CPI. In section 8 below, we show that *in the long run*, the money purchases approach will be roughly equivalent to the rental equivalence or user cost approaches, *except* that the acquisitions cost approach will lead to a CPI where housing has only about *one half* the weight that owner-occupied housing would have in the two alternative approaches. Note also that the price index that results from the application of the acquisitions approach to owner-occupied housing could be regarded as a subindex of the producer price index for the production of new dwelling units.

- **The geographic domain of definition of the index.**

⁶⁴ It can be argued that points (a), (b), (c) and (f) above are not quite consistent with the household perspective.

“A quite different aspect of HCIPs is the question of geographic coverage. This is a matter of special interest in the EU, given the fact that the Monetary Union (MU) is only a subset of the EU, and is likely to be a subset for some time, as the memberships of both the MU and the EU are likely to increase—at different rates—over the coming years.

At the heart of this question are two concepts well known to national accountants: the domestic concept and the national concept. ...

In principle, a price statistician has two choices. First, he can choose to measure the changes in prices faced by consumers normally resident in the country—in which case the prices paid by these consumers when they are outside the country also have to be included in the index. This is known as the ‘national’ concept of measurement.

Alternatively, he can choose to measure the changes in prices faced by all consumers in the country itself—in which case one must measure only domestic prices, but the weights applied must relate to the total consumption within the country, whether by the resident population or by foreign visitors. This is known as the ‘domestic’ concept of measurement.

There are both theoretical and practical aspects to this question. On a practical level, it would obviously be difficult, if not impossible, for a national price statistician to measure price changes in other countries where consumption is made by residents of his own country. In practice, he would have to use the CPIs of a range of foreign countries—many of which, of course, would not be in the EU.

But theoretically (fortunately) this approach is not called for. National inflation should surely measure national price changes, even if some of them are faced by foreign visitors.” John Astin (1999; 7).

With respect to the above problem, the HICP seems to opt for a domestic *producer* theory approach to the inclusion of transactions in the index rather than a *consumer* theory approach. The last sentence in the above quotation (which appeals to the vaguely specified notion of measuring national inflation) is completely unconvincing: if we look at price change from the viewpoint of domestic households, then foreign tourism prices are indeed relevant to the “inflation” experienced by households. (A sudden fall in the Canadian dollar certainly affects my propensity to take a winter vacation in Hawaii). Thus a properly constituted consumer price index from the viewpoint of domestic households *should* have a subindex that measures changes in foreign tourism related prices (converted to domestic currency).⁶⁵ If this tourism subindex is difficult to construct, that is another issue. At present, we are talking about the *principles* involved in CPI construction.

We summarize the above somewhat disjointed discussion as follows. The “theory” of the harmonized consumer price index seems to lack focus and an underlying firm theoretical basis. Evidently, its primary purpose is as a measure of inflation. But we are inclined to agree with Keynes that a measure of inflation based on “monetary” transactions is too broad to be useful. Thus when the inflation measurement goal of the harmonized index is narrowed down to focus on purchases of consumer goods and services in the economic territory of the Member State, the “general theory” of the HICP does not constrain the index as much as an explicit producer or consumer theory approach would. As a result, the HICP does not fit into either the consumer or producer domains of definition. However, the HICP does fulfil a useful purpose: *it provides a measure of household inflation that central bankers are interested in.*

⁶⁵ On the other hand, if we look at price change from the point of view of domestic supply (the producer point of view), then the domestic concept is the right one. However, elsewhere, the HICP takes a household or consumer point of view so there is a bit of an inconsistency here.

From the viewpoint of central bankers, the Producer Price Index (PPI) that covers sales to domestic households may not be completely satisfactory since this index excludes sales and other commodity taxes. On the other hand, the Consumer Price Index based on a Cost of Living framework has *either* user costs *or* imputed rental prices for owner occupied housing in it and both of these alternatives are unsatisfactory for central bankers. User costs have interest rates in them and bankers want to avoid targeting their own instruments. Imputed rental prices are not real transaction prices and hence may be “wrong” but in any case, they are not “real”. Thus from this central bank point of view, it makes sense to move away from a user cost approach to the treatment of durables and move toward a money purchases approach for purchases of durable consumer goods. Thus we have a justification for characteristics (a), (b) and (c) of the HICP above, which are not consistent with the COLI approach. Thus we have provided a justification for the HCIP.⁶⁶

In the earlier versions of this paper, I argued that the theoretical framework for the HICP should be revisited. Rather than using resources to further refine the present HCIP, I argued that it would be preferable to devote these resources to the construction of a *family* of consumer price indexes. One branch of this family would look at the *consumption transactions of households* (a consumer theory approach) and another branch of the family would look at *the domestic production by firms of consumer goods and services* (a producer theory approach). If characterizing these indexes by the word “economic” proved to be offensive, then this word could be dropped from the description of these indexes. What is more important is that two specific transaction domains of definition be chosen: one that looked at consumption transactions from the viewpoint of households and the other that took the producer perspective. This suggested dual approach to index number theory would help fill out the boxes in the *System of National Accounts: 1993*, where there are *basic prices* (which correspond roughly to producer prices) and *final demand prices* (which correspond to consumer prices in the case of the household components of final demand)⁶⁷. However, I now realize that it is futile to crusade against the widespread adoption of the HICP: it is too well established and it seems to fulfill a useful role for central bankers.

In section 8 below, we analyze some of the differences between the money purchases concept applied to the purchase of a durable consumer good and the user cost concept. However, before we discuss durables, we will devote the following section to a more extended discussion of point (d) above, the preference of believers in harmonized price indexes for a fixed basket formulation of the price index.

7. THE THEORY OF THE PURE PRICE AND QUANTITY INDEXES

⁶⁶ I suppose there is a central bank justification for characteristic (f) as well: a domestic central bank cannot control foreign tourist prices and hence would like to have these prices excluded from its target consumer price index.

⁶⁷ There are some problems with the *System's* methodology on the producer side; e.g., there is no user cost methodology for capital input, the role of interest is not completely recognised, the role of land, natural resources and inventories as inputs is not recognised and so on. On the consumer side, the user cost or rental equivalence approach to consumer durables is ruled out except for housing services. There is also a reluctance to make any imputations associated with the introduction of new commodities. However, the next revision of the Accounts will surely deal with these problems.

“Suppose however that, for each commodity, $Q' = Q$, then the fraction, $\Sigma(P'Q) / \Sigma(PQ)$, viz., the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, etc.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become $\Sigma(PQ') / \Sigma(PQ)$. It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indexes is identical.” Sir George H. Knibbs (1924; 43-44).

At the meeting of the Ottawa Group in Iceland, it was evident that many of the participating price statisticians were only comfortable with a concept of the price index that was based on pricing out a constant “representative” basket of commodities, $q \equiv (q_1, q_2, \dots, q_N)$, at the prices of period 0 and 1, $p^0 \equiv (p_1^0, p_2^0, \dots, p_N^0)$ and $p^1 \equiv (p_1^1, p_2^1, \dots, p_N^1)$ respectively. At the meeting, this concept was referred to as a *pure price index* and it can be seen that it corresponds to Knibbs’ (1924; 43) *unequivocal price index*. Thus the general functional form for the pure price index is

$$(35) P_K(p^0, p^1, q) \equiv p^1 \bullet q / p^0 \bullet q = \sum_{n=1}^N s_n (p_n^1 / p_n^0)$$

where the expenditure shares s_n corresponding to the quantity weights vector q are defined by:

$$(36) s_n \equiv p_n^0 q_n / p^0 \bullet q \quad \text{for } n = 1, 2, \dots, N.$$

There are at least two reasons why price statisticians might prefer the family of pure or unequivocal price indexes defined by (35):

- The fixed basket concept is easy to explain to the public.
- The formula (35) is *consistent in aggregation*; i.e., if the index is constructed in two stages, then the resulting index is equal to the index constructed in a single stage⁶⁸, a property shared by the Laspeyres and Paasche indexes (which are special cases of (35) with $q = q^0$, the base period consumption vector, or with $q = q^1$, the current period consumption vector).

The practical problem of picking q remains to be resolved and that is the problem we will address in this section.

It should be noted that Walsh (1901) (1921) also saw the price index number problem in the above framework:

“Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price-variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the

⁶⁸ Dalén (1999; 5) raises this consistency in aggregation issue. It should be noted that other (more complex) definitions of consistency in aggregation are possible: see Montgomery (1937; 40-48), Vartia (1976), Blackorby and Primont (1980) and Balk (1995; 85-86).

second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods.” Correa Moylan Walsh (1921; 90).

We will follow Walsh’s suggestion and restrict the n th quantity weight, q_n , to be an average or *mean* of the base period quantity q_n^0 and the current period quantity for commodity n q_n^1 , $m(q_n^0, q_n^1)$, for $n = 1, 2, \dots, N$.⁶⁹ Under this assumption, the pure price index (35) becomes:

$$(37) P_K(p^0, p^1, m(q^0, q^1)) \equiv \sum_{n=1}^N p_n^1 m(q_n^0, q_n^1) / \sum_{j=1}^N p_j^0 m(q_j^0, q_j^1).$$

In this section, we will restrict ourselves to strictly positive quantity vectors q^0 and q^1 and to price vectors p^0 and p^1 that are nonnegative but have at least one positive component. The mean function $m(a, b)$ is assumed to have the following two properties:

(38) $m(a, b)$ is a positive and continuous function, defined for all positive numbers a and b ; and

(39) $m(a, a) = a$ for all $a > 0$.

Property (39) is the defining property of a mean function: if the two numbers being averaged are equal to a common number, then the mean is also equal to this common number.

In order to determine the functional form for the mean function m , we shall impose some *tests* or *axioms* on the pure price index defined by (37). Let us rewrite the left hand side of (37) as $P_K(p^0, p^1, q^0, q^1)$. As in section 2, we ask that P_K satisfy the *time reversal test*, (9) above.⁷⁰ Under this hypothesis, it is immediately obvious that the mean function m must be a *symmetric mean*⁷¹; i.e., m must satisfy the following property:

(40) $m(a, b) = m(b, a)$ for all $a > 0$ and $b > 0$.

Assumption (40) still does not pin down the functional form for the pure price index defined by (37) above. For example, the function $m(a, b)$ could be the *arithmetic mean*, $(1/2)a + (1/2)b$, in which case (37) reduces to the *Marshall (1887) Edgeworth (1925) price index* P_{ME} :

$$(41) P_{ME}(p^0, p^1, q^0, q^1) \equiv p^1 \cdot [(1/2)q^0 + (1/2)q^1] / p^0 \cdot [(1/2)q^0 + (1/2)q^1].$$

The Australian statistician Knibbs preferred the above index for the following reasons:

⁶⁹ Note that we have chosen the mean function $m(q_n^0, q_n^1)$ to be independent of n .

⁷⁰ Knibbs (1924; 44) noted that the pure or unequivocal price index P_K defined by (35) satisfied the time reversal test and he enthusiastically endorsed this test: “In other words, the characteristic of reversibility applies to indexes when they are calculated in the manner indicated. It needs hardly be said that every properly deduced index must possess this characteristic.”

⁷¹ See Diewert (1993b; 361) for the properties of symmetric means.

“Again it is also self evident that the best basis of comparison is a regimen which differs the least possible amount from the actual regimens on any two dates compared. For each individual element this is, of course, the mean of the usage on the two occasions, that is $(1/2)(q_0 + q_1)$; and assuming a linear change in the quantities, there can be no better basis of comparison.” Sir George H. Knibbs (1924; 56).

On the other hand, the function $m(a,b)$ could be the *geometric mean*, $(ab)^{1/2}$, in which case (37) reduces to the *Walsh* (1901; 398) (1921; 97) *price index*, P_W ⁷²:

$$(42) P_W(p^0, p^1, q^0, q^1) \equiv \sum_{n=1}^N p_n^1 (q_n^0 q_n^1)^{1/2} / \sum_{j=1}^N p_j^0 (q_j^0 q_j^1)^{1/2} .$$

There are many other possibilities for the mean function m , including the mean of order r , $[(1/2)a^r + (1/2)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index P_K , we need to impose at least one additional test or axiom on $P_K(p^0, p^1, q^0, q^1)$.

There is a potential problem with the use of the Edgeworth Marshall price index (41) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (41), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country.⁷³ In technical terms, the Edgeworth Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1 . To prevent this problem from occurring in the use of the pure price index $P_K(p^0, p^1, q^0, q^1)$ defined by (37), we ask that P_K satisfy the following *invariance to proportional changes in current quantities test*.⁷⁴

$$(43) P_K(p^0, p^1, q^0, \lambda q^1) = P_K(p^0, p^1, q^0, q^1) \text{ for all } p^0, p^1, q^0, q^1 \text{ and all } \lambda > 0.$$

The two tests, (9) and (43), enable us to determine the precise functional form for the pure price index P_K defined by (37) above:

Proposition 6: Suppose the pure price index $P_K(p^0, p^1, q^0, q^1)$ is defined by (37) for all nonnegative but nonzero price vectors p^0, p^1 and for all strictly positive quantity vectors q^0, q^1 , where the mean function m satisfies (38) and (39). Suppose in addition that P_K satisfies the time reversal test (9) and the invariance to proportional changes in current quantities test (43). Then the pure price index P_K must be the Walsh index P_W defined by (42).

⁷² Walsh endorsed P_W as being the best index number formula: “We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance.” C.M. Walsh (1921; 103). His formula 6 is P_W defined by (42) and his 9 is the Fisher ideal defined by our (8) above. The *Walsh quantity index*, $Q_W(p^0, p^1, q^0, q^1)$ is defined as $P_W(q^0, q^1, p^0, p^1)$; i.e., the role of prices and quantities in definition (42) is interchanged. If we use the Walsh quantity index to deflate the value ratio, we obtain the implicit price index $p^1 \cdot q^1 / p^0 \cdot q^0 Q_W(p^0, p^1, q^0, q^1)$, which is Walsh’s formula 8. In the time series context, it will not matter which of Walsh’s indexes 6, 8 or 9 is used since they will approximate each other to the second order around an equal price and quantity point; see Diewert (1978a; 887-889).

⁷³ This is not likely to be a severe problem in the time series context where the change in quantity vectors going from one period to the next is small.

⁷⁴ This is the terminology used by Diewert (1992; 216). Vogt (1980) was the first to propose this test.

Thus the time reversal test (9) and the invariance test (43) serve to determine the functional form for the pure price index or Knibbs' unequivocal price index, P_K : the resulting index must be equal to Walsh's price index P_W defined by (42) above.

As Knibbs (1924; 44) noted, there is an analogous theory for the pure quantity index or the unequivocal quantity index of Knibbs. We give a brief outline of this theory. Let the pure quantity index Q_K have the following functional form:

$$(44) \quad Q_K(p^0, p^1, q^0, q^1) \equiv \sum_{n=1}^N q_n^1 m(p_n^0, p_n^1) / \sum_{n=1}^N q_n^0 m(p_n^0, p_n^1).$$

Thus on the right hand side of (44), in the numerator, the quantities of period 1, q_n^1 , are weighted by some average of the period 0 and 1 prices for the corresponding commodity, $m(p_n^0, p_n^1)$, while in the denominator, the quantities of period 0, q_n^0 , are weighted by the same average of the period 0 and 1 prices, $m(p_n^0, p_n^1)$.

Now we will restrict ourselves to strictly positive price vectors p^0 and p^1 and to quantity vectors q^0 and q^1 that are nonnegative but have at least one positive component. The mean function $m(a,b)$ is again assumed to have properties (38) and (39) above.

The meaning of the right hand side of (44) is clear: the consumption of commodity n in both periods is to be valued at a *constant* (across the two periods under consideration) reference price, say $p_n^* \equiv m(p_n^0, p_n^1)$, that is some sort of average of the prices for commodity n during those two periods, p_n^0 and p_n^1 . In the national income accounting literature, this property is known as *additivity* or *additive consistency*⁷⁵, and it is a very popular property for both national income accountants and business economists alike.

Our problem is to determine the functional form for the averaging function m if possible. To do this, we need to impose some tests or properties on the pure quantity index Q_K . As was the case with the pure price index, it is very reasonable to ask that the quantity index satisfy the *time reversal test*:

$$(45) \quad Q_K(p^1, p^0, q^1, q^0) = 1 / Q_K(p^0, p^1, q^0, q^1).$$

As was the case with the theory of the unequivocal price index, it can be seen that if the unequivocal quantity index Q_K is to satisfy the time reversal test (45), the mean function in (44) must be *symmetric*; i.e., m must satisfy (40).

We also ask that Q_K satisfy the following *invariance to proportional changes in current prices test*.

$$(46) \quad Q_K(p^0, \lambda p^1, q^0, q^1) = Q_K(p^0, p^1, q^0, q^1) \text{ for all } p^0, p^1, q^0, q^1 \text{ and all } \lambda > 0.$$

The idea behind the invariance test (46) is this: the quantity index $Q_K(p^0, p^1, q^0, q^1)$ should only depend on the *relative* prices in each period and it should not depend on

⁷⁵ In principle, the reference price for commodity n , p_n^* , could be some function $f_n(p^0, p^1)$ of all of the prices pertaining to both periods. In our formulation of Knibbs' model of the unequivocal quantity index, this more general formulation is ruled out.

the amount of inflation in either of the two periods. Another way to interpret test (46) is to look at what the test implies for the corresponding implicit price index, P_{IK} :

$$(47) P_{IK}(p^0, p^1, q^0, q^1) \equiv p^1 \bullet q^1 / p^0 \bullet q^0 Q_K(p^0, p^1, q^0, q^1).$$

If Q_K satisfies (46), then the corresponding implicit price index P_{IK} will satisfy the following *linear homogeneity property in current prices*⁷⁶:

$$(48) P_{IK}(p^0, \lambda p^1, q^0, q^1) = \lambda P_{IK}(p^0, p^1, q^0, q^1) \quad \text{for all } \lambda > 0.$$

The two tests, (45) and (46), enable us to determine the precise functional form for the pure quantity index Q_K defined by (44) above:

Proposition 7: Suppose the pure quantity index $Q_K(p^0, p^1, q^0, q^1)$ is defined by (44) for all nonnegative but nonzero quantity vectors q^0, q^1 and for all strictly positive price vectors p^0, p^1 , where the mean function m satisfies (38) and (39). Suppose in addition that Q_K satisfies the time reversal test (45) and the invariance to proportional changes in current prices test (46). Then the *pure quantity index* or Knibbs' *unequivocal quantity index* Q_K must be the Walsh quantity index Q_W ⁷⁷ defined by (49) below:

$$(49) Q_W(p^0, p^1, q^0, q^1) \equiv \sum_{n=1}^N q_n^1 (p_n^0 / p_n^1)^{1/2} / \sum_{j=1}^N q_j^0 (p_j^0 / p_j^1)^{1/2}.$$

Thus with the addition of two tests, the pure price index P_K must be the Walsh price index P_W defined by (42) and with the addition of same two tests (but applied to quantity indexes instead of price indexes), the pure quantity index Q_K must be the Walsh quantity index Q_W defined by (49). However, note that the product of the Walsh price and quantity indexes is *not* equal to the expenditure ratio, $p^1 \bullet q^1 / p^0 \bullet q^0$. Thus believers in the pure or unequivocal price and quantity index concepts *have to choose one of these two concepts*; they both cannot apply simultaneously.⁷⁸

It is interesting to note that Walsh's price index P_W defined above by (42) is a *superlative index number formula*; i.e., it is exact for the following unit cost function⁷⁹, which can provide a second order approximation to an arbitrary twice differentiable unit cost function:

$$(50) c_W(p_1, p_2, \dots, p_N) \equiv \sum_{n=1}^N \sum_{j=1}^N b_{nj} p_n^{1/2} p_j^{1/2}$$

where the b_{nj} are parameters satisfying the symmetry restrictions $b_{nj} = b_{jn}$ and some other restrictions.⁸⁰

Similarly, the Walsh quantity index defined above by (49) is also a *superlative index number formula*; i.e., it is exact for the following utility function⁸¹:

⁷⁶ This test was first proposed by Walsh (1901; 385).

⁷⁷ This is the quantity index that corresponds to the price index 8 defined by Walsh (1921; 101).

⁷⁸ Knibbs (1924) did not notice this point!

⁷⁹ See Diewert (1976; 135).

⁸⁰ See Diewert (1971). The unit cost function defined by (50) corresponds to the Generalized Leontief *production function* or, in the present context, *utility function*.

⁸¹ See Diewert (1976; 130-132) for a proof of the exactness result. The utility function defined by (51) corresponds to the Generalized Linear utility or production function defined by Diewert (1971).

$$(51) f_W(q_1, q_2, \dots, q_N) \equiv \sum_{n=1}^N \sum_{j=1}^N a_{nj} q_n^{1/2} q_j^{1/2}$$

where the a_{nj} are parameters satisfying the symmetry restrictions $a_{nj} = a_{jn}$.⁸² Thus the Walsh quantity index Q_W is both *superlative* and *additively consistent*.⁸² Thus this formula meets an objection of Hill (1993; 384), who noted that the commonly used superlative indexes (P_F defined above by (28) and P_T defined by (32)) are not additively consistent.

Some results in Diewert (1978a; 888) show that the Walsh price index P_W defined by (42) and the implicit Walsh price index defined by (47) with $Q_K = Q_W$ approximate each other to the second order around an equal price and quantity point. Thus, using normal time series data, the direct Walsh and the implicit Walsh price indexes will usually closely approximate each other (and the Fisher ideal price index as well).

Hill noted⁸³ that superlative price indexes treated the data in the two situations to be compared in a *symmetric* manner:

“Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great.” Peter Hill (1993; 384).

In this section, we have shown that the symmetric basket approach to a price index and the symmetric price weighting approach to a quantity index both lead to superlative Walsh indexes and these indexes will also closely approximate their Fisher and Törnqvist counterparts.

We turn now to a major source of difference between a cost of living index and a harmonized price index; namely the treatment of consumer durables.

8. THE MONEY PURCHASES VERSUS USER COST APPROACHES

“We have noticed also that though the benefits which a man derives from living in his own house are commonly reckoned as part of his real income, and estimated at the net rental value of his house; the same plan is not followed with regard to the benefits which he derives from the use of his furniture and clothes. It is best here to follow the common practice, and not count as part of the national income or dividend anything that is not commonly counted as part of the income of the individual.” Alfred Marshall (1898; 594-595).

As we saw in section 5 above, the treatment of consumer durables in a CPI is a contentious issue: proponents of the harmonized index tend to favor including just

⁸² Diewert (1996b; 246) noted this property of Walsh’s quantity index. Note that the additivity property applies *only* to the two periods under consideration; i.e., if the period 1 data are replaced by period 2 data, then the reference price for commodity n changes from $p_n^* = (p_n^0 p_n^1)^{1/2}$ to $p_n^{**} = (p_n^0 p_n^2)^{1/2}$.

⁸³ See also Hill (1988).

new purchases of a durable consumer good in the domain of definition of the CPI while proponents of the cost of living approach to the CPI tend to favor either a user cost or rental equivalence approach to durable goods. The differences between the three approaches are most pronounced when the durable good has a very long life, such as housing. The *rental equivalence approach*, which can be traced back to Marshall (1898; 594) at least, simply values the services yielded by the use of a consumer durable good for a period by the corresponding market rental value for the same durable for the same period of time (if such a rental value exists). This is the approach taken by the Bureau of Labor Statistics in the U.S. and in the *System of National Accounts: 1993* for owner occupied housing:

“As well-organized markets for rented housing exist in most countries, the output of own-account housing services can be valued using the prices of the same kinds of services sold on the market with the general valuation rules adopted for goods and services produced on own account. In other words, the output of housing services produced by owner-occupiers is valued at the estimated rental that a tenant would pay for the same accommodation, taking into account factors such as location, neighbourhood amenities, etc. as well as the size and quality of the dwelling itself.” Eurostat and others (1993; 134).

However, the *System of National Accounts: 1993* follows Marshall (1898; 595) and does *not* extend the rental equivalence approach to consumer durables other than housing. This seemingly inconsistent treatment of durables is explained as follows:

“The production of housing services for their own final consumption by owner-occupiers has always been included within the production boundary in national accounts, although it constitutes an exception to the general exclusion of own-account service production. The ratio of owner-occupied to rented dwellings can vary significantly between countries and even over short periods of time within a single country, so that both international and intertemporal comparisons of the production and consumption of housing services could be distorted if no imputation were made for the value of own-account services.” Eurostat and others (1993; 126).

The above reasons⁸⁴ for treating owner-occupied housing on a rental equivalence basis are certainly valid but more to the point: *purchases* of new houses simply do not *reflect the actual consumption of housing services* for the population of owner-occupiers! Thus if our purpose is to measure the real consumption of the population during a period and a price index is required to deflate nominal consumption expenditures into real consumption, then the money purchases approach to a CPI will not be satisfactory.

Since the rental equivalence approach to the treatment of consumer durables in a CPI is easy to understand, we will devote the remainder of this section to the differences between the money purchases approach and the user cost approach.

The *money purchases approach* to the treatment of consumer durables is very simple: if one unit of the good costs P^0 dollars and the reference group of households purchases q^0 units of it in period 0, then the observed total purchase cost P^0q^0 is attributed to period 0.

⁸⁴ Another reason for treating housing on a rental equivalence or user cost basis is that the money purchases approach leads to a severe underestimate of the real consumption of pensioners, who are likely to have purchased their house in a previous period. Poverty measures that exclude the services of owner-occupied housing as an imputed income component are also likely to be misleading.

The problem with this approach is that the *services* of the purchased goods are not confined to period 0. By the definition of a durable good (it lasts longer than one period), the purchase will yield a flow of services to the consumer for periods that follow period 0. Thus it does not seem appropriate to charge the *entire* purchase price P^0 to the initial period of purchase. But how should the purchase price be distributed or allocated across periods? This is a *fundamental problem of accounting*, where a similar cost allocation problem occurs when a firm purchases a durable input.⁸⁵

One solution to this cost allocation problem is the *historical cost accounting* solution, which works as follows. If the durable good lasts $T+1$ periods, then the cost accountant somehow obtains a set of $T+1$ depreciation rates, d_0, d_1, \dots, d_T , such that $d_0 + d_1 + \dots + d_T = 1$. Then $d_t P^0$ is allocated to period t for $t = 0, 1, 2, \dots, T$.

Economists have tended to take a different approach to the cost allocation problem—an approach based on opportunity costs. Thus to determine the net cost of using the durable good during period 0, we assume that one unit of the durable good is purchased at the beginning of period 0 at the price P^0 . The “used” or “second-hand” durable good can be sold at the end of period 0 at the price P_s^1 . It might seem that a reasonable net cost for the use of one unit of the consumer durable during period 0 is its initial purchase price P^0 less its end of period 0 “scrap value” P_s^1 . However, *money received at the end of the period is not as valuable as money that is received at the beginning of the period*. Thus in order to convert the end of period value into its beginning of the period equivalent value, it is necessary to *discount* the term P_s^1 by the term $1+r^0$ where r^0 is the beginning of period 0 nominal interest rate that the consumer faces. Hence we define the *period 0 user cost* u^0 for the consumer durable⁸⁶ as

$$(52) \quad u^0 \equiv P^0 - P_s^1 / (1+r^0).$$

There is another way to view the user cost formula (52): the consumer purchases the durable at the beginning of period 0 at the price P^0 and charges himself or herself the rental price u^0 . The remainder of the purchase price, I^0 , defined as

$$(53) \quad I^0 \equiv P^0 - u^0$$

is regarded as an *investment*, which is to yield the appropriate opportunity cost of capital r^0 that the consumer faces. At the end of period 0, this rate of return could be realized provided that I^0 , r^0 and the selling price of the durable at the end of the period P_s^1 satisfy the following equation:

$$(54) \quad I^0 (1 + r^0) = P_s^1.$$

⁸⁵ Ralph Turvey while commenting on an earlier version of this paper astutely noted that the purchase of some services have a similar cost allocation aspect; i.e., should the cost of an appendectomy be charged entirely to the period of purchase or should its cost be distributed over the remaining life of the consumer?

⁸⁶ This approach to the derivation of a user cost formula was used by Diewert (1974) who in turn based it on an approach due to Hicks (1946; 326).

Given P_s^1 and r^0 , (54) determines Γ^0 , which in turn, given P^0 , determines the user cost u^0 via (53)⁸⁷.

The user cost formula (52) can be put into more familiar form if we first define the period 0 *economic depreciation rate* δ and the period 0 *ex poste asset inflation rate* i^0 . Define δ by⁸⁸:

$$(55) \quad (1-\delta) \equiv P_s^1 / P^1$$

where P_s^1 is the price of a used asset at the end of period 0 and P^1 is the price of a new asset at the end of period 0. The period 0 inflation rate for the new asset i^0 is defined by:

$$(56) \quad 1+i^0 \equiv P^1 / P^0.$$

Substituting (56) into (55) gives us the following formula for the end of period 0 used asset price:

$$(57) \quad P_s^1 = (1-\delta)(1+i^0)P^0.$$

Substitution of (57) into (52) yields the following expression for the *period 0 user cost* u^0 :

$$(58) \quad u^0 = [(1+r^0) - (1-\delta)(1+i^0)]P^0 / (1+r^0) \\ = [r^0 - i^0 + \delta(1+i^0)]P^0 / (1+r^0).$$

Note that $r^0 - i^0$ can be interpreted as a period 0 *real interest rate* and $\delta(1+i^0)$ can be interpreted as an *inflation adjusted depreciation rate*.

The user cost u^0 is expressed in terms of prices that are discounted to the *beginning* of period 0. However, it is also possible to express the user cost in terms of prices that are “discounted” to the *end* of period 0. Thus define the *end of period 0 user cost* p^0 as:⁸⁹

⁸⁷ This derivation for the user cost of a consumer durable was also made by Diewert (1974; 504).

⁸⁸ The treatment of additions and renovations affects the definition of the depreciation rate δ . One way of treating this problem is to *add* these expenditures to the asset base P^0 (we assume that the initial house quantity is unity). The end of the period value of the renovated house is P_s^1 and P^1 is the end of period value for a (hypothetical) house that is one period younger and has the addition or renovation that was undertaken during the period. Formula (55) is still used in order to define the joint depreciation rate for the initial house and the renovations. This treatment means that the renovations are treated as capital expenditures and only a portion of the expenditures are attributed to the period where the expenditures took place. This treatment of renovations will tend to make the growth rate g in housing expenditures (see equation (62) below) *bigger* since renovation and addition expenditures have been growing rapidly in most developed economies.

⁸⁹ Christensen and Jorgenson (1969) derived a user cost formula similar to (40) in a different way. If the inflation rate i equals 0, then the user cost formula (40) reduces to that derived by Walras (1954; 269) (first edition 1874). This zero inflation rate user cost formula was also derived by the industrial engineer A. Hamilton Church (1901; 907-908), who perhaps drew on the work of Matheson: “In the case of a factory where the occupancy is assured for a term of years, and the rent is a first charge on profits, the rate of interest, to be an appropriate rate, should, so far as it applies to the buildings, be equal (including the depreciation rate) to the rental which a landlord who owned but did not occupy a factory would let it for.” Ewing Matheson (1910; 169), first published in 1884.

$$(59) p^0 \equiv (1+r^0)u^0 = [r^0 - i^0 + \delta(1+i^0)]P^0$$

where the last equation follows using (58).

The user cost defined by (59) can be compared to the corresponding historical cost depreciation allowance for period 0, which would be δP^0 . It can be seen that the user cost p^0 is *greater* than δP^0 by the amount of the real interest rate term, $(r^0 - i^0)P^0$, and by the inflation adjustment for the depreciation rate, $i^0 \delta P^0$.⁹⁰ In the case where the asset inflation rate i^0 is zero, the end of the period user cost defined by (59) reduces to:

$$(60) p^0 = (r^0 + \delta)P^0.$$

Again, it can be seen that the no inflation user cost p^0 is *greater* than the corresponding historical cost period 0 cost allocation, δP^0 , by the amount of the interest rate term, $r^0 P^0$. It is this difference that explains why the user cost (or rental equivalence) approach to the consumption of consumer durables will tend to give a *larger* value for consumption than the money purchases approach, as we shall see later in this section.

Abstracting from transactions costs and inflation, it can be seen that the end of the period user cost defined by (60) is an *approximate rental cost*; i.e., the rental cost for the use of a consumer (or producer) durable good should equal the opportunity cost of the capital tied up, $r^0 P^0$, plus the decline in value of the asset over the period, δP^0 . When asset inflation is brought into the picture, the situation is more complicated. As it stands, the end of the period user cost formula (59) is an *ex poste* (or after the fact) *user cost*: we cannot calculate the asset inflation rate i^0 until we have reached the end of period 0. Formula (59) can be converted into an *ex ante* (or before the fact) *user cost* formula if we interpret i^0 as an *anticipated asset inflation rate*. The resulting formula should approximate a market rental rate for the asset under inflationary conditions.

The fact that the rental rate for a consumer or producer durable good consists chiefly of foregone or imputed interest and depreciation charges can be traced back to the early industrial engineering literature:

“Machines are, in some trades, let out to hire, and a certain sum is paid for their use, in the manner of rent. This is the case amongst the frame-work knitters: and Mr. Henson, in speaking of the rate of payment for the use of their frames, states, that the proprietor receives such a rent that, besides paying the full interest for his capital, he clears the value of his frame in nine years.” Charles Babbage (1835; 287).

“No sophistry is needed to assume that these charges are in the nature of such rents, for it might easily happen that in a certain building a number of separate little shops were established, each containing one machine, all making some particular part or working on some particular operation of the same class of goods, but each shop occupied, not by a wage earner, but by an independent mechanic, who rented his space, power and machinery, and sold the finished product to the lessor. Now, in such a case, what would be the shop charges of these mechanics? Clearly they would comprise as their chief if not their only item, just the rent paid. And this rent would be made up of: (1) interest, (2) depreciation, (3)

⁹⁰ We are assuming that the real interest rate is positive and the inflation rate is nonnegative.

insurance, (4) profit on the capital involved in the building, machine, and power-transmitting and generating plant. There would also most probably be a separate charge for power according to the quantity consumed.

Exclude the item of profit, which is not included in the case of shop charge, and we find that we have approached most closely to the new plan of reducing any shop into its constituent production centres. No one would pretend that there was any insuperable difficulty involved in fixing a just rent for little shops let out on this plan.” A. Hamilton Church (1901; 907-908).

Returning to the general end of the period user cost formula (59), many price statisticians, economists and accountants have objected to the inclusion of both the interest rate term r^0 and the inflation rate (or capital gains term) i^0 in the cost of using the services of a durable for a period of time. There is a tendency in the *System of National Accounts: 1993* to regard depreciation as the only valid measure of the cost of using the services of a durable input. Thus, for example, when discussing how to measure the (constant dollar) cost of production for non-market goods and services, interest as a cost item is explicitly omitted:

“The value of the output of non-market goods and services produced by government units or non-profit institutions is estimated on the basis of the total costs incurred in their production, as explained in chapter VI. ... When it is not possible to avoid using an input measure as a proxy for an output measure, the input measure should be a comprehensive one and not confined to labour inputs. As explained below, the volume of labour inputs can be measured by compensation of employees valued at the wage and salary rates of the previous year of some fixed base year, the remuneration of each individual type of worker being revalued at the appropriate rate. The volumes of intermediate consumption, consumption of fixed capital [i.e. depreciation] and any taxes on production measured at the prices or rates of the previous year or the fixed base year should be added to obtain a comprehensive volume measure covering all inputs.” Eurostat and others (1993; 402-403).

The above quotation indicates that not only does the national accounts omit interest as a cost in using the services of a durable input but also anticipated or actual asset inflation is omitted as a benefit or negative cost item. We will not review here the theories that argue for the inclusion of these items. This review would require a rather extensive discussion.

In another part of the *System of National Accounts: 1993*, it is indicated that there are no imputed interest charges associated with the use of equity financial capital to finance the purchase of durable capital inputs:

“The amounts of rent and interest actually payable on rented land and borrowed funds are recorded in the allocation of primary income account, and the entrepreneurial income account, but the implicit rents on land owned by the enterprise and the implicit interest chargeable on the use of the enterprise’s own funds are not recorded in the accounts of the System.” Eurostat and others (1993; 175).

Thus the present system of national accounts does not have value flow categories that would allow users to find the separate components of the user costs somewhere in the accounts: for consumer durables (other than housing), the accounts list only new production and for producer durables, the accounts explicitly identify only the consumption of fixed capital.⁹¹

⁹¹ In the *System of National Accounts: 1993*, gross operating surplus is roughly the value of outputs produced during the period less intermediate inputs and labour used during the period. Net operating surplus further subtracts the consumption of fixed capital. Hill (1999b) has noted that this accounting framework can be reconciled with the user cost formula (60) if we further deduct a net (interest) return that would be imputed to the value of capital input in use.

Note that in the user cost approach to the treatment of consumer durables, the *entire* user cost formula (59) is the period 0 price. Thus in the time series context, it is *not* necessary to deflate each component of the formula *separately*; the period 0 price $p^0 \equiv [r^0 - i^0 + \delta(1+i^0)]P^0$ is compared to the corresponding period 1 price, $p^1 \equiv [r^1 - i^1 + \delta(1+i^1)]P^1$ and so on.⁹²

We now want to compare the user cost approach to the treatment of consumer durables to the money purchases approach⁹³. Obviously, in the short run the value flows associated with each approach could be very different. For example, if real interest rates, $r^0 - i^0$, are very high and the economy is in a severe recession or depression, then purchases of new consumer durables, Q^0 say, could be very low and even approach 0 for very long lived assets, like houses or autos. On the other hand, using the user cost approach, existing stocks of consumer durables would be carried over from previous periods and priced out at the appropriate user costs and the resulting consumption value flow could be quite large. Thus in the short run, the monetary values of consumption under the two approaches could be vastly different. Hence, we will restrict ourselves in what follows to a (hypothetical) longer run comparison.

Suppose that in period 0, the reference population of households purchased q^0 units of a consumer durable at the purchase price P^0 . Then *the period 0 value of consumption from the viewpoint of the money purchases approach* is:

$$(61) \quad V_M^0 \equiv P^0 q^0.$$

Recall that the end of period user cost for one new unit of the asset purchased at the beginning of period 0 was p^0 defined by (59) above. In order to simplify our analysis, we assume *declining balance depreciation*; i.e., at the beginning of period 0, a one period old asset is worth $(1-\delta)P^0$; a two period old asset is worth $(1-\delta)^2P^0$; ... ; a t period old asset is worth $(1-\delta)^tP^0$; etc. Under these hypotheses, the corresponding end of period 0 user cost for a new asset purchased at the beginning of period 0 is p^0 ; the end of period 0 user cost for a one period old asset at the beginning of period 0 is $(1-\delta)p^0$; the corresponding user cost for a two period old asset at the beginning of period 0 is $(1-\delta)^2p^0$; ... ; the corresponding user cost for a t period old asset at the

⁹² However, there is no harm in breaking up $p^0Q^0 = (r^0 + \delta)P^0Q^0$ (using formula (60) for simplicity) into the two terms, $r^0P^0Q^0$ and δP^0Q^0 , under certain conditions. The price for the first term would be r^0P^0 , the price for the second term would be δP^0 and the quantity for both terms would be Q^0 . Note that the quantities for each component would vary in strict proportion over time and thus the use of any index number formula that was consistent with Leontief's (1936) Aggregation Theorem would lead to the same aggregate results using the usual user cost approach or the separate component approach. The Paasche, Laspeyres and Fisher index number formulae are consistent with Leontief's theorem. The separate component approach may be more acceptable to users, since they could omit the parts of user cost that they were not happy with. Note also that an interest rate term is *never* deflated by itself in any of these approaches; it is always associated with a purchase price or opportunity cost price, P^0 . This observation "solves" a problem that has puzzled national income accountants: namely, how should nominal interest be deflated into real interest. In the user cost approach, a nominal interest rate always appears with an associated price.

⁹³ In Europe, this is called the net acquisitions approach.

beginning of period 0 is $(1-\delta)^t p^0$; etc.⁹⁴ Our final simplifying assumption is that household purchases of the consumer durable have been growing at the geometric rate g into the indefinite past. This means that if household purchases of the durable were q^0 in period 0, then in the previous period they purchased $q^0/(1+g)$ new units; two periods ago, they purchased $q^0/(1+g)^2$ new units; ... ; t periods ago, they purchased $q^0/(1+g)^t$ new units; etc. Putting all of these assumptions together, it can be seen that *the period 0 value of consumption from the viewpoint of the user cost approach* is:

$$(62) \quad V_U^0 \equiv p^0 q^0 + [(1-\delta)p^0 q^0/(1+g)] + [(1-\delta)^2 p^0 q^0/(1+g)^2] + \dots$$

$$= (1+g)(g+\delta)^{-1} p^0 q^0 \quad \text{summing the infinite series}$$

$$(63) \quad = (1+g)(g+\delta)^{-1} [r^0 - i^0 + \delta(1+i^0)] P^0 q^0 \quad \text{using (59).}$$

We simplify (63) by letting the asset inflation rate i^0 be 0 (so that r^0 can be interpreted as a real interest rate) and we take the ratio of the user cost flow of consumption (63) to the money purchases measure of consumption in period 0, (61):

$$(64) \quad V_U^0 / V_M^0 = (1+g)(r^0+\delta)/(g+\delta).$$

Using formula (64), it can be seen that if $1+g > 0$ and $\delta + g > 0$, then V_U^0 / V_M^0 will be greater than unity if

$$(65) \quad r^0 > g(1-\delta)/(1+g),$$

a condition that will usually be satisfied.⁹⁵ Thus under normal conditions and over a longer time horizon, *household expenditures on consumer durables using the user cost approach will tend to exceed the corresponding money outlays on new purchases of the consumer durable*. The difference between the two approaches will tend to grow as the life of the asset increases (i.e., as the depreciation rate δ decreases).

To get a rough idea of the possible magnitude of the value ratio for the two approaches, V_U^0 / V_M^0 , we evaluate (64) for a "housing" example where the depreciation rate is 2 % (i.e., $\delta = .02$), the real interest rate is 4 % (i.e., $r^0 = .04$) and the growth rate for the production of new houses is 1 % (i.e., $g = .01$). In this base case, the ratio of user cost expenditures on housing to the money outlays, V_U^0 / V_M^0 , is 2.02. If we *increase* the depreciation rate to 3 %, then V_U^0 / V_M^0 *decreases* to 1.77; if we *decrease* the depreciation rate to 1 %, then V_U^0 / V_M^0 *increases* to 2.53. Again looking at the base case, if we *increase* the real interest rate to 5 %, then V_U^0 / V_M^0 *increases* to 2.36 while if we *decrease* the real interest rate to 3 %, then V_U^0 / V_M^0 *decreases* to 1.68. Finally, if we *increase* the growth rate for new houses to 2 %, then V_U^0 / V_M^0 *decreases* to 1.53 while if we *decrease* the growth rate to 0, then V_U^0 / V_M^0 *increases* to 3.00. Thus a money outlays approach to housing in the CPI is likely to give about one half the expenditure weight that a user cost approach would give.

⁹⁴ For most consumer durables, the one hoss shay assumption for depreciation is more realistic than the declining balance model. To see the sequence of one hoss shay user costs, see Hulten (1990) and Diewert and Lawrence (1999).

⁹⁵ However, the inequality (65) is not satisfied for very rapidly growing components of consumer demand, like home computers where the growth rate might exceed 20 %. The formula for V_U^0/V_M^0 given by (64) simplifies to $1+(r^0/\delta)$, which is always greater than unity if r^0 is greater than 0, provided that the growth rate g is 0. We assume that the depreciation rate δ satisfies $0 < \delta < 1$.

Let us carry out the same sensitivity analysis for a shorter lived asset like an automobile. For this consumer durable, we take the base depreciation rate to be 15 %; i.e., we assume $\delta = .15$, $g = .01$ and $r = .04$. For this base case, the expenditure ratio for the two approaches, V_U^0 / V_M^0 defined by (64) above, reduces to 1.20. If we *increase* the depreciation rate to 20 %, then V_U^0 / V_M^0 *decreases* to 1.15; if we *decrease* the depreciation rate to 10 %, then V_U^0 / V_M^0 *increases* to 1.29. Again looking at the base case, if we *increase* the real interest rate to 5 %, then V_U^0 / V_M^0 *increases* to 1.26 (a very small increase) while if we *decrease* the real interest rate to 3 %, then V_U^0 / V_M^0 *decreases* to 1.14 (a very small decrease). Finally, if we *increase* the growth rate for new autos to 2 %, then V_U^0 / V_M^0 *decreases* to 1.14 (again a very small decrease) while if we *decrease* the growth rate to 0, then V_U^0 / V_M^0 *increases* to 1.27. *Thus a money outlays approach to autos in the CPI is likely to give about 80 % of the expenditure weight that a user cost approach would give.* This example shows that once the depreciation rate exceeds 20 %, the differences in weighting for the two approaches is likely to be small and hence the traditional money purchases approach for these shorter lived consumer durables is an acceptable approximation to a perhaps theoretically more correct user cost approach.

Let us carry out the same sensitivity analysis for a somewhat longer lived asset like furniture or household furnishings. For this consumer durable, we take the base depreciation rate to be 7 %; i.e., we assume $\delta = .07$, $g = .01$ and $r = .04$. For this base case, the expenditure ratio for the two approaches, V_U^0 / V_M^0 defined by (64) above, reduces to 1.39. If we *increase* the depreciation rate to 9 %, then V_U^0 / V_M^0 *decreases* to 1.31; if we *decrease* the depreciation rate to 5 %, then V_U^0 / V_M^0 *increases* to 1.52. Again looking at the base case, if we *increase* the real interest rate to 5 %, then V_U^0 / V_M^0 *increases* to 1.52 while if we *decrease* the real interest rate to 3 %, then V_U^0 / V_M^0 *decreases* to 1.26. Finally, if we *increase* the growth rate for new furniture to 2 %, then V_U^0 / V_M^0 *decreases* to 1.25 while if we *decrease* the growth rate to 0, then V_U^0 / V_M^0 *increases* to 1.57. *Thus a money outlays approach to furniture in the CPI is likely to give about 70 % of the expenditure weight that a user cost approach would give.*

Finally, we carry out the same sensitivity analysis for a very short lived asset like a home computer. For this consumer durable, we take the base depreciation rate to be 25 %; i.e., we assume $\delta = .25$, $g = .20$ (note the rapid assumed growth rate of 20 %) and $r = .04$. For this base case, the expenditure ratio for the two approaches, V_U^0 / V_M^0 defined by (64) above, becomes .773, which is *less* than one this time. If we *increase* the depreciation rate to 30 %, then V_U^0 / V_M^0 *increases* to .816; if we *decrease* the depreciation rate to 20 %, then V_U^0 / V_M^0 *decreases* to .720. Again looking at the base case, if we *increase* the real interest rate to 5 %, then V_U^0 / V_M^0 *increases* slightly to .800 while if we *decrease* the real interest rate to 3 %, then V_U^0 / V_M^0 *decreases* slightly to .747. Finally, if we *increase* the growth rate for new home computers to 30 %, then V_U^0 / V_M^0 *decreases* to .685 while if we *decrease* the growth rate to 10 %, then V_U^0 / V_M^0 *increases* to .911. *Thus a money outlays approach to home computers in the CPI is likely to give about 130 % of the expenditure weight that a user cost approach would give.*

The last example above shows that the ratio of the user cost flow of consumption to the money purchases measure of consumption in period 0, V_U^0 / V_M^0 , *does not always exceed unity* if the growth rate in new purchases g exceeds the real interest rate r^0 by

enough. However, when we look at all categories of consumer durables, it is virtually certain that the user cost approach will lead to higher expenditure weights for the durables category than the weights that result from the application of the money purchases approach.

We conclude this section by listing some of the problems and difficulties that might arise in implementing a user cost approach to purchases of durable consumer goods.

- It is difficult to determine what the relevant nominal interest rate r^0 is for each household. It may be necessary to simply use a benchmark interest rate that would be determined by either the government, a national statistical agency or an accounting standards board.
- It is difficult to determine what the relevant profile of depreciation rates is for each consumer durable.⁹⁶
- It will be difficult to decide on an ex ante user cost (in which case, the asset inflation rate i^0 appearing in the user cost formula (59) is a *forecasted inflation rate*) or an ex poste user cost (in which case, the asset inflation rate i^0 is the *actual asset inflation rate* over the duration of the period). The ex ante concept is appropriate for economic modelers and business forecasters while the ex poste concept is the appropriate one for measuring ex poste economic performance. Using either interpretation, there will be difficulties in forming estimates for the inflation rates.⁹⁷
- The user cost formula (59) must be generalized to accommodate various taxes that may be associated with the purchase of a durable or with the continuing use of the durable.⁹⁸

In the following section, we review some of the objections that could be directed towards the use of the cost of living concept as a guiding principle for the construction of a consumer price index.

9. CRITICISMS OF THE COST OF LIVING APPROACH

“As the Boskin report expressed it, such an index ‘is a comparison of the minimum expenditure required to achieve the same level of well-being (also known as welfare, utility, standard-of-living) across two different sets of prices.’ This concept has been expounded by a number of authors, notably Robert A. Pollak and Erwin Diewert in papers notable for their intellectual rigour, formality of expression and minimal reference to the actual behaviour of individual consumers. ...

Two questions about this theory lack an answer:

1. Whose preferences are concerned, what is a consumer? Is it a household?

⁹⁶ As mentioned earlier, it is not necessary to assume declining balance depreciation in the user cost approach: any pattern of depreciation can be accommodated, including one hoss shay depreciation, where the durable yields a constant stream of services over time until it is scrapped. See Diewert and Lawrence (1999) for some empirical examples for Canada using different assumptions about depreciation.

⁹⁷ Using the ex poste interpretation, the difficulty will be in determining the profile of used asset prices at the beginning and end of each period. For additional material on the difficulties involved in constructing user costs, see Diewert (1980; 475-486). For empirical comparisons of different user cost formulae, see Harper, Berndt and Wood (1989) and Diewert and Lawrence (1999).

⁹⁸ For example, property taxes are associated with the use of housing services.

2. How can this static, timeless, theory be applied to a period of time? What is its appropriate length? Presumably it must be short enough for prices to remain constant throughout, but long enough for a consumer to buy the set of items.” Ralph Turvey (1999; 1-2).

There is no shortage of criticisms of the economic approach to the determination of a consumer price index. In this section, we will list some of these criticisms (and add some of our own) and respond as best we can to them. Hopefully, this listing will inspire other researchers to overcome some of these problems with the theory of the cost of living index.

- **The price of commodity n for household h in period t is not well defined**

Within each period, the household may make many purchases of a commodity. It is unlikely (unless the time period is very short, in which case, most household purchases will be at the zero level) that every purchase will be made at the same price. Thus we have to ask: exactly what *is* the period t price vector $p^t \equiv (p_1^t, p_2^t, \dots, p_N^t)$, which appeared in equation (1) above? Diewert (1995b), following Walsh (1901; 96) (1921; 88) and Davies (1924), argued that perhaps the best choice for p_n^t is the period t *unit value* (total value divided by total quantity) for commodity n, calculated over the appropriate transactions domain of definition. It should be noted that the other approaches to index number theory face the same problem in defining price and quantity at the lowest level of aggregation.

- **Prices are not constant across households**

In a world of sales and vigorous retail competition, this criticism will certainly be true.⁹⁹ How can we patch up the theory outlined in section 2 above in order to take into account the possibility that prices for a commodity may not be constant across households?

Define the price faced by household h for commodity n and the quantity consumed in period t by p_{hn}^t and q_{hn}^t respectively. Define the *total consumption* across all households for commodity n in period t by q_n^t :

$$(66) \quad q_n^t \equiv \sum_{h=1}^H q_{hn}^t \quad ; n = 1, \dots, N ; t = 0, 1.$$

The corresponding aggregated over households average price for commodity n in period t must be the *unit value* p_n^t defined as:

$$(67) \quad p_n^t \equiv \sum_{h=1}^H p_{hn}^t q_{hn}^t / q_n^t \quad ; n = 1, \dots, N ; t = 0, 1.$$

Having defined the above individual components of market prices and quantities, define the *aggregate period t price and quantity vectors* as:

$$(68) \quad p^t \equiv (p_1^t, \dots, p_N^t) \quad ; t = 0, 1;$$

$$(69) \quad q^t \equiv (q_1^t, \dots, q_N^t) \quad ; t = 0, 1.$$

⁹⁹ Note also that prices will not be constant across households for the same commodity if the government supplies or subsidises certain commodities (e.g., housing or medical services) conditional on the income or wealth status of the household.

Define also the *individual household h price and quantity vectors for period t* as:

$$(70) \quad p_h^t \equiv (p_{h1}^t, \dots, p_{hN}^t) \quad ; t = 0, 1; h = 1, \dots, H;$$

$$(71) \quad q_h^t \equiv (q_{h1}^t, \dots, q_{hN}^t) \quad ; t = 0, 1; h = 1, \dots, H.$$

It is easy to verify that the aggregate price and quantity vectors, p^t and q^t defined by (68) and (69), satisfy

$$(72) \quad p^t \bullet q^t = \sum_{h=1}^H p_h^t \bullet q_h^t \quad t = 0, 1.$$

Now insert the household specific price vectors into the definition of the family of theoretical price indexes defined by (2) in section 2 above and we obtain the following definition for the *theoretical price index* P^* :

$$(73) \quad P^*(p_1^0, \dots, p_H^0, p_1^1, \dots, p_H^1, u, e_1, e_2, \dots, e_H) \equiv \sum_{h=1}^H C^h(u_h, e_h, p_h^1) / \sum_{h=1}^H C^h(u_h, e_h, p_h^0).$$

Since prices are no longer assumed to be constant across households, in definition (73), there H vectors of household prices for period 0, p_1^0, \dots, p_H^0 , and H vectors of household prices for period 1, p_1^1, \dots, p_H^1 , instead of just the market price vectors, p^0 and p^1 , in definition (2).

The old inequality (6) now becomes:

$$(74) \quad P^*(p_1^0, \dots, p_H^0, p_1^1, \dots, p_H^1, u^0, e_1^0, e_2^0, \dots, e_H^0) \\ \equiv \sum_{h=1}^H C^h(u_h^0, e_h^0, p_h^1) / \sum_{h=1}^H C^h(u_h^0, e_h^0, p_h^0) \\ = \sum_{h=1}^H C^h(u_h^0, e_h^0, p_h^1) / \sum_{h=1}^H p_h^0 \bullet q_h^0 \quad \text{using (1) for } t = 0 \\ \leq \sum_{h=1}^H p_h^1 \bullet q_h^0 / \sum_{h=1}^H p_h^0 \bullet q_h^0$$

since $C^h(u_h^0, e_h^0, p_h^1) \equiv \min_q \{ p_h^1 \bullet q : f^h(q, e_h^0) \geq u_h^0 \} \leq p_h^1 \bullet q_h^0$ and q_h^0 is feasible for the cost minimization problem for $h = 1, 2, \dots, H$

$$(75) \quad = \sum_{h=1}^H p_h^1 \bullet q_h^0 / p^0 \bullet q^0 \quad \text{using (72) for } t = 0 \\ \equiv P_{DL}$$

where P_{DL} is defined to be the *disaggregated (over households) Laspeyres price index*, $\sum_{h=1}^H p_h^1 \bullet q_h^0 / \sum_{h=1}^H p_h^0 \bullet q_h^0$, which uses the individual vectors of household quantities for period 0, (q_1^0, \dots, q_H^0) , as quantity weights. In a similar fashion, the old inequality (7) becomes:

$$(76) \quad P^*(p_1^0, \dots, p_H^0, p_1^1, \dots, p_H^1, u^1, e_1^1, e_2^1, \dots, e_H^1) \\ \equiv \sum_{h=1}^H C^h(u_h^1, e_h^1, p_h^1) / \sum_{h=1}^H C^h(u_h^1, e_h^1, p_h^0) \\ = \sum_{h=1}^H p_h^1 \bullet q_h^1 / \sum_{h=1}^H C^h(u_h^1, e_h^1, p_h^0) \quad \text{using (1) for } t = 1 \\ \geq \sum_{h=1}^H p_h^1 \bullet q_h^1 / \sum_{h=1}^H p_h^0 \bullet q_h^1 \quad \text{using a feasibility argument} \\ = p^1 \bullet q^1 / \sum_{h=1}^H p_h^0 \bullet q_h^1 \quad \text{using (72) for } t = 1 \\ (77) \quad \equiv P_{DP}$$

where P_{DP} is defined to be the *disaggregated (over households) Paasche price index*, $\sum_{h=1}^H p_h^1 \bullet q_h^1 / \sum_{h=1}^H p_h^0 \bullet q_h^1$, which uses the individual vectors of household quantities for period 1, (q_1^1, \dots, q_H^1) , as quantity weights.

Using the inequalities, (74) and (76), it is possible to modify the proof of Proposition 1 and prove the following result:

Proposition 8: Under the assumptions of Proposition 1, there exists a reference utility vector $u^* \equiv (u_1^*, u_2^*, \dots, u_H^*)$ such that the household h reference utility level u_h^* lies between the household h period 0 and 1 utility levels, u_h^0 and u_h^1 respectively for $h = 1, \dots, H$, and there exist household environmental vectors $e_h^* \equiv (e_{h1}^*, e_{h2}^*, \dots, e_{hM}^*)$ such that the household h reference m th environmental variable e_{hm}^* lies between the household h period 0 and 1 levels for the m th environmental variable, e_{hm}^0 and e_{hm}^1 respectively for $m = 1, 2, \dots, M$ and $h = 1, \dots, H$, and the conditional plutocratic cost of living index $P^*(p_1^0, \dots, p_H^0, p_1^1, \dots, p_H^1, u^*, e_1^*, \dots, e_H^*)$, defined by (73) evaluated at this intermediate reference utility vector u^* and the intermediate reference vector of household environmental variables (e_1^*, \dots, e_H^*) , lies between the observable (in principle) disaggregated Laspeyres and Paasche price indexes, P_{DL} and P_{DP} , defined above by (75) and (77).¹⁰⁰

With prices no longer assumed to be equal across households, the disaggregated Laspeyres price index P_{DL} defined by (75) will no longer necessarily equal the usual aggregate Laspeyres price index, $P_L \equiv p^1 \bullet q^0 / p^0 \bullet q^0$, where p^0 , p^1 and q^0 are defined above by (66)-(69). Similarly, the disaggregated Paasche price index P_{DP} defined by (77) will no longer necessarily equal the usual aggregate Paasche price index, $P_P \equiv p^1 \bullet q^1 / p^0 \bullet q^1$, where p^0 , p^1 and q^1 are also defined above by (66)-(69). Since it is much, much easier to evaluate the aggregate Paasche and Laspeyres indexes than their disaggregated counterparts, it will be useful to determine under what conditions P_L will equal P_{DL} and when P_P will equal P_{DP} . We now address this problem.

For later reference, define the (arithmetic) *average household consumption* of commodity n in period t by:

$$(78) \quad q_{An}^t \equiv \sum_{h=1}^H (1/H) q_{hn}^t \quad ; t = 0, 1 ; n = 1, \dots, N.$$

Similarly, define the (arithmetic) *average household price* for commodity n in period t as:

$$(79) \quad p_{An}^t \equiv \sum_{h=1}^H (1/H) p_{hn}^t \quad ; t = 0, 1 ; n = 1, \dots, N.$$

Comparing the aggregate Laspeyres price index, $P_L \equiv p^1 \bullet q^0 / p^0 \bullet q^0$, with the disaggregated Laspeyres index P_{DL} defined by (75), it can be seen that the denominators for each index are the same. Thus we need only compare their numerators. We start with the terms in the numerator of the disaggregated Laspeyres index P_{DL} that involve commodity n and add and subtract some terms:

¹⁰⁰ If each household consumption vector in period 1 is proportional to its period 0 consumption vector, so that $q_h^1 = \lambda q_h^0$ for $h = 1, \dots, H$ (note that the proportionality factor λ is constant across households), then $P_{DL} = P_{DP}$ and the theoretical index P^* described in Proposition 8 is equal to this common value. Of course, if household prices are proportional, so that $p_h^1 = \lambda p_h^0$ for $h = 1, \dots, H$ (note that the proportionality factor λ is constant across households), then the theoretical index P^* is equal to the common proportionality factor λ .

$$\begin{aligned}
(80) \quad \sum_{h=1}^H p_{hn}^1 q_{hn}^0 &= \sum_{h=1}^H p_n^1 q_{hn}^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) q_{hn}^0 \\
&= p_n^1 q_n^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) q_{hn}^0 && \text{using (66) for } t = 0 \\
&= p_n^1 q_n^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) (q_{hn}^0 - q_{An}^0) + \sum_{h=1}^H (p_{hn}^1 - p_n^1) q_{An}^0 \\
&&& \text{using (78) for } t = 0 \\
&= p_n^1 q_n^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) (q_{hn}^0 - q_{An}^0) + H (p_{An}^1 - p_n^1) q_{An}^0 \\
&&& \text{using (79) for } t = 1.
\end{aligned}$$

Looking at the right hand side of (80), we see that the second set of terms is a (modified)¹⁰¹ *covariance* between the prices of commodity n that households faced in period 1 and their consumption of commodity n in period 0. Of course, this covariance term and the last term in (80) vanish if all households face the same price for commodity n in period 1. In the general case, the sign of this covariance is unknown but its magnitude is likely to be small. The last term on the right hand side of (80) is proportional to the difference $p_{An}^1 - p_n^1$ between the arithmetic mean of the commodity n household prices in period 1, p_{An}^1 , and the market quantity weighted average price or unit value for commodity n in period 1, p_n^1 . Now if each household demanded a positive quantity of each of the N consumer goods and services in period 1, we would probably feel confident in asserting that the difference in these two average prices for commodity n , $p_{An}^1 - p_n^1$, is likely to be positive, since we would expect households who face below average prices for the commodity to purchase more of it. Thus the arithmetic average of the prices, p_{An}^1 , will tend to exceed the weighted average, p_n^1 , and the last term on the right hand side of (80) will be *positive*. However, if N is large, so that we have a very fine disaggregation of commodities, then the situation is very different. In this case, there will be many households that do not consume a positive amount of each commodity in period 1. Thus if household h consumed *no* units of commodity n in period 1, it is clear that q_{hn}^1 equals 0 *but the corresponding price must be taken to be the Hicksian (1940) reservation price* p_{hn}^1 *that will just cause the household to demand 0 units of commodity* n *in period 1*. This reservation price p_{hn}^1 will tend to be *below* the market unit value for commodity n in period 1, p_n^1 . Thus if the number of commodities distinguished is large, we would expect the last term on the right hand side of (80) to be *negative*. Summing all of this up, it can be seen that we are uncertain as to the sign and magnitude of the last two sets of terms on the right hand side of (80).

Using (80) for each commodity n , it can be seen that we obtain the following relationship between the two Laspeyres indexes:

$$\begin{aligned}
(81) \quad P_{DL} &= P_L + [\sum_{n=1}^N \sum_{h=1}^H (p_{hn}^1 - p_n^1) (q_{hn}^0 - q_{An}^0) / p^0 \bullet q^0] \\
&\quad + H [\sum_{n=1}^N (p_{An}^1 - p_n^1) q_{An}^0 / p^0 \bullet q^0].
\end{aligned}$$

¹⁰¹ It is not quite a covariance because the arithmetic average of the period 1 commodity n prices, p_{An}^1 , is replaced by the period 1 weighted average or market unit value for commodity n , p_n^1 . This technique of comparing two different weighted averages of prices using a covariance or a correlation coefficient dates back to Bortkiewicz (1923; 374-375).

Of course, it will be difficult to evaluate empirically the last two terms on the right hand side of (81) due to the difficulties involved in estimating reservation prices for commodities that households did not consume in period 1.

The above analysis can be duplicated in order to find a relationship between the aggregate Paasche price index, $P_P \equiv p^1 \bullet q^1 / p^0 \bullet q^1$, and the disaggregated Paasche index P_{DL} defined by (77). By symmetry, the counterpart to (80) is

$$(82) \quad \sum_{h=1}^H p_{hn}^0 q_{hn}^1 = p_n^0 q_n^1 + \sum_{h=1}^H (p_{hn}^0 - p_n^0)(q_{hn}^1 - q_{An}^1) + H (p_{An}^0 - p_n^0) q_{An}^1.$$

Again, we are uncertain as to the sign and magnitude of the last two sets of terms on the right hand side of (82). We can use the relations (82) for each commodity n and we obtain the following Paasche counterpart to (81), which we write in the following convenient form:

$$(83) \quad 1/P_{DL} = [1/P_L] + [\sum_{n=1}^N \sum_{h=1}^H (p_{hn}^0 - p_n^0)(q_{hn}^1 - q_{An}^1) / p^1 \bullet q^1] \\ + H [\sum_{n=1}^N (p_{An}^0 - p_n^0) q_{An}^1 / p^1 \bullet q^1].$$

As was the case with the last two terms on the right hand side of (81), it is uncertain what the sign and magnitude of the last two terms on the right hand side of (83) are.¹⁰²

It is possible to obtain a somewhat different relationship between the terms in the numerator of the disaggregated Laspeyres index P_{DL} that involve commodity n and the corresponding commodity n term in the aggregated Laspeyres index, $p_n^1 q_n^0$. Repeating the first two lines of (80), we have:

$$(84) \quad \sum_{h=1}^H p_{hn}^1 q_{hn}^0 = \sum_{h=1}^H p_n^1 q_{hn}^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) q_{hn}^0 \\ = p_n^1 q_n^0 + \sum_{h=1}^H (p_{hn}^1 - p_n^1) q_{hn}^0 \\ = p_n^1 q_n^0 + \sum_{h=1}^H p_{hn}^1 q_{hn}^0 - p_n^1 H q_{An}^0 \quad \text{using (78)} \\ = p_n^1 q_n^0 + \sum_{h=1}^H p_{hn}^1 q_{hn}^0 - [\sum_{h=1}^H p_{hn}^1 q_{hn}^1 / H q_{An}^1] H q_{An}^0 \\ \quad \text{using (67) for } t = 1 \\ = p_n^1 q_n^0 + q_{An}^0 \sum_{h=1}^H p_{hn}^1 ([q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1]) \\ = p_n^1 q_n^0 + q_{An}^0 \sum_{h=1}^H (p_{hn}^1 - p_{An}^1) ([q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1]) \\ \quad + q_{An}^0 \sum_{h=1}^H p_{An}^1 ([q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1]) \\ = p_n^1 q_n^0 + q_{An}^0 \sum_{h=1}^H (p_{hn}^1 - p_{An}^1) ([q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1])$$

since $\sum_{h=1}^H q_{hn}^0 / q_{An}^0 = H$ and $\sum_{h=1}^H q_{hn}^1 / q_{An}^1 = H$ as well. Looking at the right hand side of (84), we see that the last set of terms is q_{An}^0 times the inner product of a vector of deviations from the average price p_{An}^1 of household prices p_{hn}^1 for commodity n in period 1 with a vector of (scaled) differences in the consumption of commodity n by households over the two periods, $[q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1]$, $h = 1, \dots, H$. If household quantities consumed for commodity n are *proportional* for the two periods, so that $(q_{1n}^1, \dots, q_{Hn}^1) = \lambda_n (q_{1n}^0, \dots, q_{Hn}^0)$, then this last vector of quantity differences will be 0

¹⁰² We note that the theory of the producer price index that was outlined in section 4 above could be reworked using the techniques in this section if *firms* faced different prices for the N commodities. For a given firm and a given commodity, supply prices could differ due to price discrimination (by the selling firm) between the various outlets that it supplies or due to changes in costs between deliveries.

and the last set of terms on the right hand side of (84) will be 0 as well. Similarly, if all the household prices for commodity n are *identical* in period 1, so that $p_{An}^1 = p_{hn}^1$ for $h = 1, \dots, H$, then the last set of terms on the right hand side of (84) will be 0 as well. These two conditions are *sufficient* for the terms $\sum_{h=1}^H p_{hn}^1 q_{hn}^0$ in the disaggregated Laspeyres formula to equal the corresponding commodity n term in the aggregate Laspeyres formula $p_n^1 q_n^0$. A *necessary and sufficient* condition for the equality of these two sets of commodity n terms is that the vector of period 1 price deviations from equality have a 0 inner product with the vector of deviations from proportionality of the two (scaled) quantity vectors for commodity n .

Obviously, equations (84) may be used for each commodity n and it can be seen that we obtain the following relationship (analogous to (81) above) between the two Laspeyres indexes:

$$(85) P_{DL} = P_L + \sum_{n=1}^N \{q_{An}^0 \sum_{h=1}^H (p_{hn}^1 - p_{An}^1) ([q_{hn}^0 / q_{An}^0] - [q_{hn}^1 / q_{An}^1]) / p^0 \bullet q^0\}.$$

As was the case when we evaluated the likely magnitude of the last two terms on the right hand side of (81), we are not quite sure what the sign and magnitude of the last set of terms on the right hand side of (85) will be in empirical applications.¹⁰³

Proposition 8 above implies that from a theoretical point of view, it does not matter all that much if households do not face the same prices for commodities in each period: the same old theory works and the new theoretical index could be approximated by the Fisher ideal index using the disaggregated Laspeyres and Paasche indexes, $(P_{DL}P_{DP})^{1/2}$. However, from an empirical point of view, there are some problems:

- We know that the aggregate Paasche and Laspeyres indexes tend to be numerically close to each other if the periods being compared are close and hence taking a symmetric mean of the two indexes is likely to provide a good point approximation to the underlying theoretical index. We have not built up the same empirical experience using the disaggregated Paasche and Laspeyres indexes so we are uncertain as to how close to each other they will be.
- If we disaggregate commodities very finely, we encounter the zero demand problem for individual households and then we have to estimate reservation prices—a very perilous project indeed. In other words, in a world of finely disaggregated commodities, we simply will not have the primary information that is required to evaluate the disaggregated “economic” Paasche and Laspeyres indexes.

In practice, in a world where the number of commodities is large, we will have to content ourselves with evaluating the usual Fisher ideal index, $(P_L P_P)^{1/2}$, which makes use of aggregate data, and hope that the difference between the aggregated and disaggregated Fisher indexes is small.

- **Households do not face prices that are independent of the quantity purchased**

Situations that fit into this criticism include:

¹⁰³ We leave it to the reader to derive the Paasche counterparts to the Laspeyres equations (84) and (85).

- Price discounts for bulk purchases or frequent buyer discounts that do not involve a fixed cost for joining the “club”;
- The after tax price of leisure (the after tax wage rate) changes as more hours are worked due to a progressive income tax;
- Frequent buyer discounts that are contingent on paying a membership fee;
- Discounts for “tied” purchases of other commodities.

In all of the above situations, the price that the household faces for a commodity is not completely independent of the quantities supplied or demanded by the household. Actually, Frisch (1936; 14-15) showed how, in theory, the economic approach can be adapted to deal with prices that are dependent on quantities purchased or supplied. What is required to implement his approach is a knowledge of the nonlinear budget set that a household actually faces, taking into account the dependence of prices on quantities. If these nonlinear budget sets are known and differentiable at the point where the household ends up in each period¹⁰⁴, then the nonlinear budget set can be linearized and the coefficient associated with each quantity can be used in place of its price and the analysis can proceed along the same lines as in the subsection above, where households faced different prices for the same commodity. However, the chances of a national statistical office actually implementing such a complicated approach seem rather remote at the moment.

- **Household composition is not constant over time**

All of our theories of a plutocratic cost of living thus far have assumed that *exactly* the same households are present in the two periods being compared. In reality, marriages take place, children become adults, there are deaths and there is in and out migration. Thus the households that are present in period 0 will not be *exactly* identical to the households that are present in period 1.

This is indeed a problem with the cost of living index theory that we have presented in section 2 and in the material immediately above. We can only suggest two solutions to this problem:

- Ignore the problem. Typically, when making comparisons over periods that are reasonably close in time, household composition will not change very much.
- Try to make adjustments to the aggregate data to exclude households that were present in one period but not in the other. Obviously, if complete micro data on each household were available, this would not be a problem and we could restrict our comparison to households that were more or less unchanged and present in both periods.

We pass on to the next set of criticisms of the economic approach to the consumer price index.

¹⁰⁴ See Diewert (1993a; 171). The technique can be adapted to the nondifferentiable case as well; see Wales and Woodland (1976) (1979). The same linearization techniques can be applied to the theory of the producer price index when the firm has market power; see Diewert (1993a; 172) and Paul (1999; 149-160).

- **Household preferences and environmental variables are not constant**

There are many specific criticisms that fall under this general heading, including:

- Tastes are not constant. In particular, education and general life experience systematically changes ones tastes and preferences from period to period.¹⁰⁵
- Biological aging also systematically changes one tastes. In particular, as one becomes very old, choice sets become more restricted due to deterioration in physical skills and in mental acuity.
- Accidents and illness also impact upon choice sets that are feasible for consumers going from period to period; i.e., a severe accident effectively changes ones feasible preferences going from one period to the next.
- Exogenous environmental variables (such as the weather or temperature) could be different in the two periods and these differences could change consumer preferences.

In response to criticisms of this type, we revised the preliminary version of this paper so that the model presented in section 2 above now accommodates changing environmental variables. One of the environmental variables could be time t , which could be used as a variable to map the preferences of period 0 into the preferences of period 1 in a continuous manner; i.e., household h 's utility function could be defined as $f^h(q,t)$, a continuous function, with $f^h(q,0)$ representing the preferences of period 0 and $f^h(q,1)$ representing the preferences of period 1. Thus taste changes and gradual aging could be accommodated using the model of consumer behavior presented in section 2. However, the model cannot readily accommodate discontinuous or discrete changes in tastes or environmental variables. This is an area that requires further thought and research.

- **The assumption that the household has well defined preferences over all possible commodities is unrealistic**

There is some considerable merit in this criticism. Think first of a multiple adult household. How are consistent household preferences to be formed from individual preferences? This is not a trivial problem. Even in the case of a single person household, how is the individual to even know about all of the millions of possible consumer goods and services that are out there somewhere, let alone form consistent preferences over these commodities? However, if commodity n is not consumed by household h in both periods under consideration, then it can be *dropped* from

¹⁰⁵ Balk (1989) deals with the problem of changing tastes as follows. Neglecting environmental variables, consider the case of a single consumer who has the direct utility function $f^t(q)$ in period t for $t = 0,1$. Let $C^t(u,p)$ be the corresponding dual cost functions for $t = 0,1$. Then Balk defines his Laspeyres perspective true cost of living index as $P_B^0 \equiv C^1(f^1(q^0), p^1)/C^0(f^0(q^0), p^0) = C^1(f^1(q^0), p^1)/p^0 \bullet q^0$ where q^0 solves the period 0 cost minimisation problem. Thus Balk compares the base period expenditures, $p^0 \bullet q^0$, with the minimum cost $C^1(f^1(q^0), p^1)$ (using the new preferences) of achieving the utility level $f^1(q^0)$, which is the utility that the base period quantity vector q^0 yields in period 1 using the new preference function f^1 , given that the consumer faces the period 1 price vector, p^1 . A simple feasibility argument shows that this new index satisfies the usual Laspeyres index bounding inequality, $P_B^0 \leq p^1 \bullet q^0 / p^0 \bullet q^0 \equiv P_L$. Balk also defines his Paasche perspective true cost of living index as $P_B^1 \equiv C^1(f^1(q^1), p^1)/C^0(f^0(q^1), p^0) = p^1 \bullet q^1 / C^0(f^0(q^1), p^0)$ where q^1 solves the period 1 cost minimisation problem. A simple feasibility argument shows that this new index satisfies the usual Paasche index bounding inequality, $P_B^1 \geq p^1 \bullet q^1 / p^0 \bullet q^1 \equiv P_p$. These new theoretical indexes seem very sensible to me.

household h 's utility function and the preference map can be restricted to the much smaller set of commodities that are actually consumed in at least one of the two periods. For commodities that are consumed repeatedly over many periods, it is at least plausible that consistent household preferences over these commodities might emerge. However, consistency problems are likely to arise when we consider how an individual forms preferences over “new” commodities that were not tried in previous periods. Advertising, marketing of new products, the experience of friends, reading magazines that rate new products—all of these factors will influence preference formation over new goods and problems of inconsistency could arise. At least economists are willing to explore these problems of preference determination and how to measure the benefits of new products whereas the approach of (most) price statisticians has been to simply ignore new commodities in their price indexes.

- **Traditional consumer theory ignores the problems posed by household production**

Peter Hill, in discussing the provocative study by Nordhaus (1997) on the price of light, has raised the issue as to how should a cost of living index treat household production where consumers combine purchased market goods or “inputs” to produce finally demanded “commodities” that yield utility:

“There is another area in which the definition of a COL requires further clarification and precision. From what is utility derived? Households do not consume many of the goods and services they purchase directly but use them to produce other goods or services from which they derive utility. In a recent stimulating and important paper, Nordhaus has used light as a case study. Households purchase items such as lamps, electric fixtures and fittings, light bulbs and electricity to produce light, which is the product they consume directly. ... The light example is striking because Nordhaus provides a plausible case for arguing that the price of light, measured in lumens, has fallen absolutely (at least in US dollars) and dramatically over the last two centuries as a result of major inventions, discoveries and ‘tectonic’ improvements in the technology of producing light.

The question that arises is whether goods and services that are essentially *inputs* into the production of other goods and services should be treated in a COL as if they provided utility directly. In principle, a COL should include the shadow, or imputed, prices, of the outputs from these processes of production and not the prices of the inputs. ... There is a need to clarify exactly how this issue is to be dealt with in a COL index.” Peter Hill (1999a; 5).

We attempt to clarify the issues raised by Hill by using the model of household production of finally demanded commodities that was postulated by Becker (1965) many years ago. Becker’s model illustrates not only how household production of the type mentioned by Hill can be integrated into a cost of living framework, but it also indicates the important role that the *allocation of household time* plays in a realistic model of household behavior. In Becker’s model of consumer behavior, a household (consisting of a single individual for simplicity) purchases q_n units of *market commodity* n and combines it with a household input of time, t_n , to produce $z_n = f_n(q_n, t_n)$ units of a *finally demanded commodity* for $n = 1, 2, \dots, N$ say, where f_n is *the household production function* for the n th finally demanded commodity¹⁰⁶. During the period of time under consideration, the household also offers t_L hours of time on the labor market, earning an after tax wage of w per hour. The consumer-worker has preferences over different combinations of the finally demanded commodities and hours of work that are summarized by the *utility function*, $U(z_1, \dots, z_N, t_L)$. In addition

¹⁰⁶ More complicated household production functions could be introduced but the present assumptions suffice to show how household production can be modelled in a COLI framework.

to the budget constraint, the household has to satisfy the *time constraint*, $\sum_{n=1}^N t_n + t_L = T$, where T is the number of hours available in the period under consideration. Rather than study the consumer's utility maximization problem subject to the budget and time constraints, we will study the equivalent consumer's cost or expenditure minimization problem subject to a utility constraint plus the time constraint. Thus we assume that the observable consumption vector $(q_1^0, \dots, q_N^0) \equiv q^0$, time allocation vector $(t_1^0, \dots, t_N^0) \equiv t^0$ and labor supply t_L^0 solve the following period 0 expenditure minimization problem:

$$(86) \min_{q^0 \text{'s and } t^0 \text{'s}} \{ \sum_{n=1}^N p_n^0 q_n^0 - w^0 t_L^0 : U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), t_L] = u^0; \sum_{n=1}^N t_n + t_L = T \}$$

where $u^0 \equiv U[f_1(q_1^0, t_1^0), \dots, f_N(q_N^0, t_N^0), t_L^0]$ is the utility level actually attained by the household in period 0, $(p_1^0, \dots, p_N^0) \equiv p^0$ is the vector of commodity prices that the household faces in period 0 and w^0 is the after tax wage rate faced by the consumer-worker in period 0.

If we use the time constraint in (86) to eliminate the hours worked variable t_L , we obtain an equivalent period 0 expenditure minimization problem and we find that under the above assumptions, q^0 and t^0 solve:

$$(87) \min_{q^0 \text{'s and } t^0 \text{'s}} \{ \sum_{n=1}^N [p_n^0 q_n^0 + w^0 t_n^0] - w^0 T : U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), T - \sum_{n=1}^N t_n] = u^0 \}.$$

Now we introduce a simpler notation for the utility function, treating the vector of time allocation variables $t = (t_1, \dots, t_N)$ as a vector of environmental variables:

$$(88) f(q, t) = f(q_1, \dots, q_N, t_1, \dots, t_N) \equiv U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), T - \sum_{n=1}^N t_n].$$

Thus q^0 and t^0 also solve:

$$(89) \min_{q^0 \text{'s and } t^0 \text{'s}} \{ \sum_{n=1}^N [p_n^0 q_n^0 + w^0 t_n^0] - w^0 T : f(q, t) = u^0 \}.$$

Now if we condition on the optimal time allocation variables, $t^0 \equiv (t_1^0, \dots, t_N^0)$, it can be seen that $q^0 \equiv (q_1^0, \dots, q_N^0)$ solves:

$$(90) \min_q \{ \sum_{n=1}^N p_n^0 q_n^0 : f(q, t^0) = u^0 \} \equiv C(u^0, p^0, t^0)$$

where $C(p, t)$ is the conditional cost function that corresponds to the utility function $f(q, t)$. In a similar fashion, letting $t^1 \equiv (t_1^1, \dots, t_N^1)$ be the optimal vector of time allocation variables for the consumer-worker's period 1 expenditure minimization problem that is analogous to (86), we can show that the consumer's observed period 1 consumption vector $q^1 \equiv (q_1^1, \dots, q_N^1)$ solves:

$$(91) \min_q \{ \sum_{n=1}^N p_n^1 q_n^1 : f(q, t^1) = u^1 \} \equiv C(u^1, p^1, t^1)$$

where $u^1 \equiv f(q^1, t^1)$. Now we can more or less repeat the analysis presented in section 2 above, with the time variables in the vector t replacing the environmental variables in the vector e . Thus we can define a theoretical family of cost of living indexes,

$$(92) P^*(p^0, p^1, u, t) \equiv C(u, p^1, t) / C(u, p^0, t)$$

that is indexed by the utility level u and the vector of time variables $t \equiv (t_1, \dots, t_N)$. As usual, we can specialize u and t to equal the period 0 utility level u^0 and the vector of period 0 time allocations, t^0 , and we can derive the Laspeyres upper bound:

$$(93) \quad P^*(p^0, p^1, u^0, t^0) \leq p^1 \bullet q^0 / p^0 \bullet q^0 \equiv P_L.$$

We can also specialize u to equal the period 1 utility level u^1 and t to equal the period vector of period 1 time allocations, t^1 , and we can derive the usual Paasche lower bound:

$$(94) \quad P^*(p^0, p^1, u^1, t^1) \geq p^1 \bullet q^1 / p^0 \bullet q^1 \equiv P_P.$$

Finally, we can adapt the proof of Proposition 1 and show that there exists a reference utility level u^* that lies between the period 0 and 1 utility levels, u^0 and u^1 , and a reference time allocation vector t^* whose components lie between the period 0 and 1 time allocation vectors, t^0 and t^1 , such that $P^*(p^0, p^1, u^*, t^*)$ lies between the observable Laspeyres and Paasche indexes for our consumer-worker, P_L and P_P .

Thus a theory of the cost of living index that is based on a model where consumers buy market goods and combine them, along with time inputs, to yield (unobservable) finally demanded commodities is completely isomorphic to the theory of the conditional cost of living index, where time variables take the place of the environmental variables. There is no need to estimate shadow prices for these finally demanded commodities.

Some points of interest emerge from the above analysis:

- If we want to base our theory for the consumer price index on *an unconditional cost function*, say $C^*(u, p, w)$ where $C^*(u^0, p^0, w^0)$ is the optimized objective function for (86), then it will be necessary to collect information on the household's allocation of time.
- The utility function $f(q, t)$ defined by (88) above is a *blend* of the consumer's *utility function* U defined over finally demanded commodities z_n and labor supply t_L , $U(z_1, \dots, z_N, t_L)$, and the *household production functions*, $f_n(q_n, t_n)$, $n = 1, \dots, N$. Thus the *blended utility function* $f(q, t)$ will not remain constant over time due to *technological progress* in the production of finally demanded commodities, as in the Nordhaus light analysis. Hence shifts in the blended utility function f over time could be due to taste changes or to production innovations.

Unfortunately, the household production story is actually more complicated than we have indicated in the above model for many households: the above analysis neglects the production of market goods and services by households. High rates of income taxation in many industrialized countries may have stimulated increased household production of goods and services that are either immediately or eventually put on the marketplace.¹⁰⁷ The stimulus for this home production is that in many cases, the labor

¹⁰⁷ William Watson (1999) explains the problem as follows: "I spent Labour Day, fittingly, at work. ... I was scraping my front porch and filling the holes with wood filler, in preparation for painting it ... Objectively speaking, the reason I found myself scraping and patching was taxes. My comparative advantage, as we economists say, is typing, not hand tools. I should really be paying someone else to

effort at home is not taxed. As a result, households demand not only traditional consumer goods like food and drink but also nontraditional producer goods like home computers (used for self employment production), office supplies and building materials for renovations.¹⁰⁸

The above criticism applies to all consumer price index approaches since it is really a *domain of definition problem*: do we want to measure *only* the consumption of households during a period (and have a separate set of accounts for household production of market goods and services) or do we want to *combine* consumption with home production? In any case, the economic approach can be adapted (in theory at least) to deal with either domain of definition. In the *pure consumption approach*, we need to partition all household purchases into purchases that are directed towards consumption alone and into purchases that are inputs into the household production function. We also need to allocate household time into time spent on consumption activities, time spent on household production and time spent on external work (and commuting to work).¹⁰⁹ Then only household consumption related purchases would appear in the domain of definition of the consumer price index. In the combined consumption and household production approach, household consumption and market production related purchases would appear in the domain of definition of the consumer price index.

We now indicate how the above model of a consumer-worker who supplied only labor services on the marketplace (this was the time allocation t_L) could be generalized to the case where the worker also devotes some time to home production of marketable goods and services (this will be the time allocation t_H).

We assume that the vector of inputs used in the home production function for marketable commodities is $Q_I \equiv (Q_{I1}, \dots, Q_{Ij})$ and the vector of outputs produced is $Q_O \equiv (Q_{O1}, \dots, Q_{OK})$. The corresponding vectors of market prices for inputs and outputs in period t are $P_I^t \equiv (P_{I1}^t, \dots, P_{Ij}^t)$ and $P_O^t \equiv (P_{O1}^t, \dots, P_{OK}^t)$ respectively for $t = 0, 1$. Given the availability of the vector of market inputs Q_I and given that the home worker is to produce the vector of market outputs Q_O , then we assume that the minimum amount of time that is required to implement this home production plan is $t_H = G(Q_I, Q_O)$. The function G is a *factor requirements function* and it summarizes the home production technology for the external marketplace. Our earlier utility function for finally demanded commodities, $U(z_1, \dots, z_N, t_L)$ must now be generalized to allow for the relative disutility of working at home production t_H hours versus working in the marketplace t_L hours. Thus our new final utility function is $U(z_1, \dots, z_N, t_L, t_H)$. As above, we continue to assume that the finally demanded commodities are produced by combining market purchases of consumer goods and services (the q_n) with household time (the t_n) according to the final demand production functions, $z_n = f_n(q_n, t_n)$ for $n =$

paint the front porch. The reason I don't is taxes. Taxes mean I have to pay roughly four times what the job is worth. First, because my marginal rate is 50 plus per cent, I have to earn twice as much in pre-tax income as a painter would charge me. And, depending on the painter's income tax rate and GST status, he has to charge me close to twice what he wants in after-tax income. Two times two being four (even in Tax-land), to pay for the job, I end up having to earn four times what the folks I would hire think their time is worth."

¹⁰⁸ This last good has some consumer good characteristics since the renovations may lead to increased enjoyment around the house but many renovations are undertaken for business purposes, since capital gains on owner-occupied houses are often tax exempt.

¹⁰⁹ To see the importance of time allocation information, see Becker (1965).

1, ..., N. The new time constraint is $\sum_{n=1}^N t_n + t_L + t_H = T$, where T is the number of hours available in the period under consideration, $\sum_{n=1}^N t_n$ is the total number of hours spent on the production of "home" finally demanded commodities, t_L is the number of hours spent working on the external labor market at the wage rate w and t_H is the number of hours spent working at home producing goods and services to be sold on the market. We now assume that the observable consumption vector $(q_1^0, \dots, q_N^0) \equiv q^0$, input vector $(Q_{I1}^0, \dots, Q_{IN}^0) \equiv Q_I^0$, market output vector $(Q_{O1}^0, \dots, Q_{OK}^0) \equiv Q_O^0$, time allocation vector $(t_1^0, \dots, t_N^0) \equiv t^0$, labor supply t_L^0 and time spent on home production for the external marketplace t_H^0 solve the following period 0 expenditure minimization problem:

$$(95) \min_{q^0, s, Q^0, t^0} \{ \sum_{n=1}^N p_n^0 q_n^0 - w^0 t_L^0 + \sum_{j=1}^J P_{Ij}^0 Q_{Ij}^0 - \sum_{k=1}^K P_{Ok}^0 Q_{Ok}^0 : \\ U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), t_L, t_H] = u^0; \sum_{n=1}^N t_n + t_L + t_H = T; t_H = G(Q_I, Q_O) \}$$

where $u^0 \equiv U[f_1(q_1^0, t_1^0), \dots, f_N(q_N^0, t_N^0), t_L^0, t_H^0]$ is the utility level actually attained by the household in period 0, $(p_1^0, \dots, p_N^0) \equiv p^0$ is the vector of commodity prices that the household faces in period 0, $P_I^0 \equiv (P_{I1}^0, \dots, P_{IN}^0)$ is the period 0 price vector for the inputs into the home production function, $P_O^0 \equiv (P_{O1}^0, \dots, P_{OK}^0)$ is the period 0 price vector for the outputs produced by work at home and w^0 is the after tax market wage rate faced by the consumer-worker in period 0. If we use the time constraint in (95) to eliminate the hours worked variable t_L and use the home production constraint $t_H = G(Q_I, Q_O)$ to eliminate t_H , we obtain an equivalent period 0 expenditure minimization problem and we find that under the above assumptions, q^0, Q_I^0, Q_O^0 and t^0 solve:

$$(96) \min_{q^0, s, Q^0, t^0} \{ \sum_{n=1}^N [p_n^0 q_n^0 + w^0 t_n^0] - w^0 T + \sum_{j=1}^J P_{Ij}^0 Q_{Ij}^0 - \sum_{k=1}^K P_{Ok}^0 Q_{Ok}^0 : \\ U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), T - \sum_{n=1}^N t_n - G(Q_I, Q_O), G(Q_I, Q_O)] = u^0 \}.$$

Now we introduce a simpler notation for the utility function, treating the vector of time allocation variables $t = (t_1, \dots, t_N)$ as a vector of environmental variables:

$$(97) f(q, Q_I, Q_O, t) \equiv U[f_1(q_1, t_1), \dots, f_N(q_N, t_N), T - \sum_{n=1}^N t_n - G(Q_I, Q_O), G(Q_I, Q_O)].$$

Since the utility function f has absorbed the home production function G into it (as well as the final demand production functions f_1, \dots, f_N), we shall call f the household's *home production utility function*. Using our new notation for the utility function, it can be seen that q^0, Q_I^0, Q_O^0 and t^0 solve:

$$(98) \min_{q^0, s, Q^0, t^0} \{ \sum_{n=1}^N [p_n^0 q_n^0 + w^0 t_n^0] - w^0 T + P_I^0 \bullet Q_I - P_O^0 \bullet Q_O : f(q, Q_I, Q_O, t) = u^0 \}.$$

Now if we condition on the optimal time allocation variables, $t^0 \equiv (t_1^0, \dots, t_N^0)$, it can be seen that $q^0 \equiv (q_1^0, \dots, q_N^0)$, $(Q_{I1}^0, \dots, Q_{IN}^0) \equiv Q_I^0$, $(Q_{O1}^0, \dots, Q_{OK}^0) \equiv Q_O^0$ solves:

$$(99) \min_{q^0, s, Q^0} \{ \sum_{n=1}^N p_n^0 q_n^0 + P_I^0 \bullet Q_I - P_O^0 \bullet Q_O : f(q, Q_I, Q_O, t^0) = u^0 \} \equiv C(u^0, p^0, P_I^0, P_O^0, t^0)$$

where $C(u^0, p, P_I, P_O, t)$ is the *conditional home production cost function* that corresponds to the home production utility function $f(q, Q_I, Q_O, t)$. In a similar fashion, letting $t^1 \equiv (t_1^1, \dots, t_N^1)$ be the optimal vector of time allocation variables for the consumer-worker's period 1 expenditure minimization problem that is analogous to (95), we can show that the consumer's observed period 1 consumption vector $q^1 \equiv$

(q_1^1, \dots, q_N^1) , period 1 home production input demand vector $(Q_{I1}^1, \dots, Q_{IN}^1) \equiv Q_I^1$ and period 1 home production output vector $(Q_{O1}^1, \dots, Q_{OK}^1) \equiv Q_O^1$ solves:

$$(100) \min_{q^1, s, Q^1} \{ \sum_{n=1}^N p_n^1 q_n^1 + P_I^1 \bullet Q_I^1 - P_O^1 \bullet Q_O^1 : f(q, Q_I, Q_O, t^1) = u^1 \} \equiv C(u^1, p^1, P_I^1, P_O^1, t^1)$$

where $u^1 \equiv f(q^1, Q_I^1, Q_O^1, t^1)$. Now we can repeat the analysis presented immediately above. Thus we can define a theoretical family of cost of living indexes,

$$(101) P^*(p^0, P_I^0, P_O^0, p^1, P_I^1, P_O^1, u, t) \equiv C(u, p^1, P_I^1, P_O^1, t) / C(u, p^0, P_I^0, P_O^0, t)$$

that is indexed by the utility level u and the vector of time variables $t \equiv (t_1, \dots, t_N)$. Note that the home production vectors of input and output prices for the two periods under consideration now appear in definition (101) along with the usual consumer commodity price vectors p^0 and p^1 . This is as it should be since we have combined home production with consumption. Hence, if the components of the output price vector P_O *increase* or the components of the input price vector P_I *decrease*, then the minimum cost of achieving the utility level u , $C(u, p, P_I, P_O, t)$, will *decrease*, which is beneficial for our consumer-worker.

As usual, we can specialize u to equal u^0 and t to equal the period 0 vector of time allocations, t^0 , and we can derive the following Laspeyres upper bound:

$$(102) P^*(p^0, P_I^0, P_O^0, p^1, P_I^1, P_O^1, u^0, t^0) \leq [p^1 \bullet q^0 + P_I^1 \bullet Q_I^0 - P_O^1 \bullet Q_O^0] / [p^0 \bullet q^0 + P_I^0 \bullet Q_I^0 - P_O^0 \bullet Q_O^0] \\ \equiv P_L$$

We can also specialize u to equal the period 1 utility level u^1 and t to equal the period vector of period 1 time allocations, t^1 , and we can derive the following Paasche lower bound¹¹⁰:

$$(103) P^*(p^0, P_I^0, P_O^0, p^1, P_I^1, P_O^1, u^1, t^1) \geq [p^1 \bullet q^1 + P_I^1 \bullet Q_I^1 - P_O^1 \bullet Q_O^1] / [p^0 \bullet q^1 + P_I^0 \bullet Q_I^1 - P_O^0 \bullet Q_O^1] \\ \equiv P_P.$$

Finally, we can adapt the proof of Proposition 1 and show that there exists a reference utility level u^* that lies between the period 0 and 1 utility levels, u^0 and u^1 , and a reference time allocation vector t^* whose components lie between the period 0 and 1 time allocation vectors, t^0 and t^1 , such that $P^*(p^0, P_I^0, P_O^0, p^1, P_I^1, P_O^1, u^*, t^*)$ lies between the observable Laspeyres and Paasche indexes for our consumer-worker, P_L and P_P .

The implications of the above model of home production for the construction of a consumer price index seem to be rather significant. For the self employed who work at home, production and consumption are completely intertwined. Thus it will be difficult to separate out the usual consumption vector q from the input demand vector Q_I which is used to produce market outputs Q_O . However, the alternative to separation of the two types of activity (consumption and market production) is the

¹¹⁰ We assume that the numerators and denominators in (101) and (102) are positive.

above rather complex model, which most price statisticians will probably regard as being unrealistic. In any case, the above issues deserve more attention.¹¹¹

We turn to the next criticism of the economic approach to consumer price indexes.

- **The economic approach assumes that goods can be purchased in fractional units instead of integral amounts**

Diewert noted this problem and proposed a solution¹¹²:

“Most goods can only be purchased in integral numbers, and for most goods, this does not cause major problems. However, some durable goods such as cars and houses may be purchased only in integer units, and such purchases would form a large share of the consumer’s total expenditure. Hence we cannot neglect the lumpiness problem for such classes of durables. How may we apply traditional ‘continuous’ utility and index number theory to this situation? ... For all practical purposes, we can replace the original preferences defined over integer combinations of TV sets by continuous preferences with ‘kinks’. The resulting preference function $F(x)$ may be treated in the normal manner as far as index number theory is concerned. Note that the economic effect of the ‘kinks’ will be to make the consumer change his durable holdings only after relatively large changes in the rental prices of the durables relative to nondurable goods; i.e., responses will be ‘sticky’. This point should be taken into account in econometric work, but it need not concern us from the viewpoint of index number theory.” W.E. Diewert (1983a; 212-213).

Thus the problem of integer purchases does not present a major challenge to the economic approaches to index number theory and so we pass on to our last criticism of the economic approach.

- **Economic approaches to the CPI do not deal adequately with the problem of seasonal commodities**

The above criticism is actually a criticism that applies equally well to *all* approaches to index number theory. The problem is this: a seasonal commodity can be present in one month or quarter and then be absent from the marketplace in the following month or quarter. How then are we to calculate the price change pertaining to the commodity over the two periods when the commodity is simply not present in one of the periods? It is simply a mission impossible to do this!

Turvey (1979) conducted an ingenious experiment to see if the presence of seasonal commodities in a CPI could be a problem empirically. He constructed an artificial data set giving fictitious monthly price and quantity data for 5 types of fruit for 4 years. He sent this data set to every statistical agency in the world with the instructions to construct a monthly price index using this data and using their normal seasonal adjustment procedures. Needless to say, the answers varied tremendously.

The problems raised by Turvey remain with us today. Diewert (1999) has recently taken a new look at this very old problem from the perspective of the economic approach to index number theory. Diewert concluded that in the presence of seasonal

¹¹¹ In addition to the favourable tax treatment of home production and self employment income, the internet is making it possible for white collar workers to work at home. Thus for many countries, self employment is increasing.

¹¹² Technically, we replace the original set of commodity combinations that can yield at least the utility level u by its convex free disposal hull.

commodities, there is a need for at least *three* separate consumer price indexes. The *first index* should be a short term month to month index defined over *nonseasonal commodities*.¹¹³ This index should be useful for the purpose of monitoring short run inflationary trends in the economy. The *second index* should be a year over year index, where the prices in January are compared to the January prices of a base year, the prices in February are compared to the February prices of a base year, etc. This index should give an accurate measure of year over year inflation, which is free from seasonal influences. The *third index* should be an annual one¹¹⁴, which compares a moving total of 12 months with 12 base year months. This type of annual index can serve as a substitute for the present classes of seasonally adjusted price indexes that rely on “black box” time series methods for seasonal adjustment.

In any case, the topic of seasonal adjustment deserves a lot more attention in the CPI literature than it has received in recent years by price statisticians.

10. CONCLUSION

“Those who wish to argue for including the welfare gains from new goods within the scope of a COL index cannot be allowed to occupy the theoretical high ground by contending that this is what economic theory requires. Economic theory does not dictate the domain of an index and it is not true that broadly defined (and heterogeneous) COL indexes are *inherently* superior to narrower (and more homogeneous) COL indexes. The domain depends on the intended use of the index. Most users of consumer price indexes are not interested in changes in a COL which are attributable to factors such as climatic changes, political events, or even scientific and technological progress. They are interested in changes in the cost of living attributable to changes in the prices of goods and services actually purchased by households.” Peter Hill (1999a; 7).

This paper started out with the objective of comparing different types of index number to see how they would serve certain uses. Along the way, the paper focused on the technical differences between a *harmonized* CPI (which is to measure consumer price inflation) and a CPI based on either consumer or producer theory, which we termed *economic* indexes.

The main differences between the two types of index are as follows:

- Functional form differences.
- Domain of definition differences.
- Differences in the treatment of consumer durables.
- Differences in the treatment of new goods and services.

With respect to functional forms for the price index, the economic approaches in sections 2 and 3 above ended up picking out the Fisher ideal price index, PF defined by (8) above, as “best”. On the other hand, in the pure theory of the harmonized index expounded in section 7 above, we found that the Walsh price index, PW defined by (42) was the theoretically “best” choice. However, these indexes approximate

¹¹³ Since harmonized indexes are supposed to exclude new commodities from their domains of definition due to the difficulties in making objective and reproducible comparisons, it would seem that harmonized indexes should also exclude seasonal commodities on the same grounds.

¹¹⁴ This index can be built up from the second class of year over year indexes.

each other to the second order around an equal price and quantities point so that for normal time series data, the numerical values of the two indexes will be very, very close. Thus the theoretical differences between the two approaches with respect to the choice of functional form is small.¹¹⁵

With respect to domain of definition differences, we saw in section 6 above that harmonized indexes do not have a clear interpretation as either a measure of household price change or of domestic producer price change for consumer goods and services. While recognizing the right of countries to choose any domain of definition for their consumer price index that they wish, we would prefer that the resources that are going into the EU harmonized indexes be diverted into developing a more complete system of household and firm price indexes. Ideally, this more complete system would have all of the information that is contained in the current harmonized indexes but also a great deal of useful additional information.

With respect to differences in the treatment of consumer durables, there are some large differences between the consumer side economic approach (where a rental equivalence or user cost approach seems appropriate) and the harmonized approach (which either omits some durables altogether or uses a money purchases approach). In section 8, we showed that in the long run, the difference between the user cost approach and the money purchases approach to the treatment of consumer durables boils down to different expenditure weighting for the two approaches. Again, it seems reasonable that *statistical agencies give out enough information so that both approaches are made available to the public.*

The final set of major differences between the harmonized and economic approaches has to do with the treatment of new goods and services: most harmonizers want to exclude new products from their indexes and make no imputations for shadow prices and the like, since these imputations are not likely to be *objective* and *reproducible*. On the other hand, most proponents of the economic approach note that a great deal of scientific, engineering and marketing effort is presently going into the development of new products and ignoring this effort will lead to a *very erroneous picture of both welfare change and productivity change*. In order to recognize the valid concerns of both camps, it would be useful for statistical agencies to perhaps provide *two* sets of indexes for each domain of definition. One index would be of the harmonized type with no imputations or quality change adjustments. The other index would make imputations and do quality adjustments. Again, the public would be well served by providing more complete information rather than just giving one approach or the other. This strategy would meet the objections of Hill noted at the beginning of this section.

APPENDIX: PROOFS OF SELECTED PROPOSITIONS

Proof of Proposition 1:

¹¹⁵ Of course, harmonizers tend to favour the Laspeyres index on *practical* grounds. However, as we argued in section 7, the Lloyd-Moulton formula could be used to approximate P_F or P_W very closely so we are not sure about the validity of these practical concerns.

Define $g(\lambda)$ for $0 \leq \lambda \leq 1$ by $g(\lambda) \equiv P^*(p^0, p^1, (1-\lambda)u^0 + \lambda u^1, (1-\lambda)e^0 + \lambda e^1)$. Note that $g(0) = P^*(p^0, p^1, u^0, e^0)$ and $g(1) = P^*(p^0, p^1, u^1, e^1)$. There are 24 possible a priori inequality relations that are possible between the four numbers $g(0)$, $g(1)$, P_L and P_P . However, the inequalities (6) and (7) above imply that $g(0) \leq P_L$ and $P_P \leq g(1)$. This means that there are only six possible inequalities between the four numbers:

- (A1) $g(0) \leq P_L \leq P_P \leq g(1)$;
- (A2) $g(0) \leq P_P \leq P_L \leq g(1)$;
- (A3) $g(0) \leq P_P \leq g(1) \leq P_L$;
- (A4) $P_P \leq g(0) \leq P_L \leq g(1)$;
- (A5) $P_P \leq g(1) \leq g(0) \leq P_L$;
- (A6) $P_P \leq g(0) \leq g(1) \leq P_L$.

Using the assumptions that: (a) each household's utility function f^h is continuous over its domain of definition; (b) each utility function is subject to local nonsatiation and (c) the price vectors p^l have strictly positive components, it is possible to use Debreu's (1959; 19) Maximum Theorem (see also Diewert (1993a; 112-113) for a statement of the Theorem) to show that household cost functions $C^h(u_h, e_h, p^l)$ will be continuous in the variables u_h, e_h for each household. Thus using definition (2), it can be seen that $P^*(p^0, p^1, u, e)$ will also be continuous in the components of the vectors u and e . Hence $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between $g(0)$ and $g(1)$. By inspecting the inequalities (A1)-(A6) above, it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_L \leq g(\lambda^*) \leq P_P$ for case (A1) or such that $P_P \leq g(\lambda^*) \leq P_L$ for cases (A2) to (A6). Now define $u^* \equiv (1-\lambda^*)u^0 + \lambda^*u^1$ and $e^* \equiv (1-\lambda^*)e^0 + \lambda^*e^1$ and the proof is complete.

Proof of Proposition 2: See Diewert (1997; 138).

Proof of Proposition 6: Assume that the number of commodities N is greater than one. We have already noted that the time reversal test (9) implies that the mean function m must satisfy the symmetry property (40). Substitution of (37) into the invariance test (43) yields the following equation, which must be valid for all $p^0 > 0_N$, $p^1 > 0_N$, $q^0 \gg 0_N$, $q^1 \gg 0_N$ and $\lambda > 0$:

$$[\sum_{i=1}^N p_i^1 m(q_i^0, \lambda q_i^1)] [\sum_{j=1}^N p_j^0 m(q_j^0, q_j^1)] = [\sum_{i=1}^N p_i^1 m(q_i^0, q_i^1)] [\sum_{j=1}^N p_j^0 m(q_j^0, \lambda q_j^1)] \text{ or}$$

$$(A7) \sum_{i=1}^N \sum_{j=1}^N p_i^1 [m(q_i^0, \lambda q_i^1) m(q_j^0, q_j^1) - m(q_i^0, q_i^1) m(q_j^0, \lambda q_j^1)] p_j^0 = 0.$$

Set all components of p^1 equal to 0 except the first component, p_1^1 , which we set equal to 1. Set all components of p^0 equal to 0 except the second component, p_2^0 , which we set equal to 1. Then (A7) becomes:

$$(A8) m(q_1^0, \lambda q_1^1) m(q_2^0, q_2^1) - m(q_1^0, q_1^1) m(q_2^0, \lambda q_2^1) = 0.$$

Let $a \equiv q_1^0$, $b \equiv q_1^1$, $c \equiv q_2^0$, $d \equiv q_2^1$. Then using these definitions and the positivity property of m , (38), after some rearrangement, (A8) becomes:

$$(A9) m(a, \lambda b) / m(a, b) = m(c, \lambda d) / m(c, d).$$

The equation (A9) holds for all positive a , b , c , d and λ . Now as a and b vary, the right hand side of (A9) remains constant. Hence the left hand side of (A9) must also be constant as a and b vary and so there exists a positive function of one variable, $f(\lambda)$ say, such that for all positive a , b and λ :

$$(A10) \quad m(a, \lambda b) / m(a, b) = f(\lambda).$$

Hence for all $a > 0$, $b > 0$ and $\lambda > 0$, we have:

$$(A11) \quad m(a, \lambda b) = f(\lambda)m(a, b).$$

Substituting $a = 1$ and $b = 1$ into (A10) yields:

$$(A12) \quad \begin{aligned} f(\lambda) &= m(1, \lambda 1) / m(1, 1) \\ &= m(1, \lambda) \end{aligned} \quad \text{using (39) which implies } m(1, 1) = 1.$$

Substituting (A12) back into (A11) yields:

$$(A13) \quad m(a, \lambda b) = m(1, \lambda)m(a, b).$$

Now set $a = 1$ in (A13) and using (A12), the resulting equation is:

$$(A14) \quad f(\lambda b) = f(\lambda)f(b) \quad \text{for all } \lambda > 0 \text{ and } b > 0.$$

Since $f(b) = m(1, b)$, using (38), f is a continuous function of one variable. But (A14) is one of Cauchy's (1821) functional equations (see Eichhorn (1978; 3) for a more recent reference) and under our assumptions on the mean function m , has the solution:

$$(A15) \quad f(\lambda) = \lambda^c \text{ for some constant } c \neq 0.$$

In order to determine m , set $b = 1$ and evaluate (A13):

$$(A16) \quad \begin{aligned} m(a, \lambda) &= m(1, \lambda)m(a, 1) \\ &= m(1, \lambda)m(1, a) && \text{using the symmetry property (40) for } m \\ &= f(\lambda)f(a) && \text{using (A12) above.} \end{aligned}$$

Substitution of (A15) into (A16) yields the following functional form for m :

$$(A17) \quad m(a, b) = a^c b^c \quad \text{for all } a > 0 \text{ and } b > 0.$$

Finally, set $a = b$ in (A17) and obtain

$$(A18) \quad m(a, a) = a^{2c} = a \quad \text{using (39).}$$

The second equality in (A18) implies $c = 1/2$ and substituting this value for c back into (A17) gives us the functional form for m ; i.e., $m(a, b) = a^{1/2} b^{1/2}$.

Proof of Proposition 7: Analogous to the proof of Proposition 6.

REFERENCES

- Allen, R.G.D. (1949), "The Economic Theory of Index Numbers", *Economica, New Series* 16, 197-203.
- Astin, J. (1999), "The European Union Harmonized Indices of Consumer Prices", paper tabled at the 5th Meeting of the Ottawa Group on Price Indices, Reykjavik, Iceland, August 25-27, 1999.
- Archibald, R.B. (1977), "On the Theory of Industrial Price Measurement: Output Price Indexes", *Annals of Economic and Social Measurement* 6, 57-72.
- Babbage, C. (1835), *On the Economy of Machinery and Manufactures*, Fourth Edition, London: Charles Knight.
- Balk, B.M. (1989), "Changing Consumer Preferences and the Cost-of-Living Index: Theory and Nonparametric Expressions", *Journal of Economics* 50, 157-169.
- Balk, B.M. (1995), "Axiomatic Price Index Theory", *International Statistical Review* 63, 69-93.
- Balk, B.M. (1998), *Industrial Price, Quantity and Productivity Indices*, Boston: Kluwer Academic Publishers.
- Becker, G.S. (1965), "A Theory of the Allocation of Time", *The Economic Journal* 75, 493-517.
- Berglund, A. (1999), "New Inflation Measure Used as Main Indicator in the ECB/ESCB Monetary Policy for the Euro-Zone", paper presented at the Conference on the Measurement of Inflation, Cardiff University, Conference Organisers: Mick Silver and David Fenwick, August 31-September 1.
- Blackorby, C. and D. Primont (1980), "Index Numbers and Consistency in Aggregation", *Journal of Economic Theory* 22, 87-98.
- Bortkiewicz, L.v. (1923), "Zweck und Struktur einer Preisindexzahl", *Nordisk Statistisk Tidsskrift* 2, 369-408.
- Bowley, A.L. (1899), "Wages, Nominal and Real", pp. 640-651 in *Dictionary of Political Economy*, Volume 3, R.H.I. Palgrave (ed.), London: Macmillan.
- Bowley, A.L. (1901), *Elements of Statistics*, Westminster: Orchard House.
- Bowley, A.L. (1919), "The Measurement of Changes in the Cost of Living", *Journal of the Royal Statistical Society* 82, 343-361.
- Cauchy, A.L. (1821), *Cours d'analyse de l'École Polytechnique*, Volume 1, *Analyse algébrique*, Paris.
- Caves, D.W., L.R. Christensen and W.E. Diewert (1982), "The Economic Theory of Index Numbers and the Measurement of Input, Output and Productivity", *Econometrica* 50, 1393-14-14.
- Christensen, L.R. and D.W. Jorgenson (1969), "The Measurement of U.S. Real Capital Input, 1929-1967," *Review of Income and Wealth* 15(4): 293-320.
- Church, A. H. (1901), "The Proper Distribution of Establishment Charges, Part III", *The Engineering Magazine* 21, 904-912.
- Dalén, J. (1992), "Computing Elementary Aggregates in the Swedish Consumer Price Index", *Journal of Official Statistics* 8, 129-147.
- Dalén, J. (1999), "Some Issues in CPI Construction", paper presented at the 5th Meeting of the Ottawa Group on Price Indices, Reykjavik, Iceland, August 25-27, 1999.
- Davies, G.R. (1924), "The Problem of a Standard Index Number Formula", *Journal of the American Statistical Association* 19, 180-188.

- Debreu, G. (1959), *The Theory of Value*, New York: John Wiley and Sons.
- Diewert, W.E. (1971), "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function", *Journal of Political Economy* 79, 481-507.
- Diewert, W.E. (1974), "Intertemporal Consumer Theory and the Demand for Durables," *Econometrica* 42, 497-516 .
- Diewert, W.E. (1976), "Exact and Superlative Index Numbers," *Journal of Econometrics* 4, 115-146; reprinted as pp.223-252 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1978a), Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900; reprinted as pp. 253-273 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1978b), "Hicks' Aggregation Theorem and the Existence of a Real Value added Function", pp. 17-51 in *Production Economics: A Dual Approach to Theory and Applications, Volume 2*, M. Fuss and D. McFadden (eds.), Amsterdam: North-Holland; reprinted as pp. 435-470 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1980), "Aggregation Problems in the Measurement of Capital", pp. 433-528 in *The Measurement of Capital*, Dan Usher (ed.), University of Chicago Press, Chicago.
- Diewert, W.E. (1981), "The Economic Theory of Index Numbers: A Survey", pp. 163-208 in *Essays in the Theory and Measurement of Consumer Behaviour in Honour of Sir Richard Stone*, A. Deaton (ed.), London: Cambridge University Press; reprinted as pp. 177-221 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1983a), "The Theory of the Cost of Living Index and the Measurement of Welfare Change", pp. 163-233 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada, reprinted as pp. 79-147 in *Price Level Measurement*, W.E. Diewert (ed.), Amsterdam: North-Holland, 1990.
- Diewert, W.E. (1983b), "The Theory of the Output Price Index and the Measurement of Real Output Change", pp. 1049-1113 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.
- Diewert, W.E. (1987), "Index Numbers", pp. 767-780 in *The New Palgrave: A Dictionary of Economics*, Volume 2, J. Eatwell, M. Milgate and P. Newman (eds.), Macmillan, London; reprinted as pp. 71-104 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248; reprinted as pp. 317-353 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.
- Diewert, W.E. (1993a), "Duality Approaches to Microeconomic Theory", pp. 105-175 in *Essays in Index Number Theory, Volume 1*, W. E. Diewert and A. O. Nakamura (eds.), North-Holland, Amsterdam.

- Diewert, W. E. (1993b), "Symmetric Means and Choice Under Uncertainty", pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W. E. Diewert and A. O. Nakamura (eds.), North-Holland, Amsterdam.
- Diewert, W.E. (1993c), "The Early History of Price Index Research", pp. 33-65 in *Essays in Index Number Theory*, Volume 1, W. E. Diewert and A. O. Nakamura (eds.), North-Holland, Amsterdam.
- Diewert, W.E. (1995a), "On the Stochastic Approach to Index Numbers", Discussion Paper 95-31, Department of Economics, University of British Columbia, Vancouver, Canada, September.
- Diewert, W.E. (1995b), "Axiomatic and Economic Approaches to Elementary Price Indexes", Discussion Paper 95-01, Department of Economics, University of British Columbia, Vancouver, Canada, January.
- Diewert, W. E. (1996a), "Comment on CPI Biases", *Business Economics* 31:2, 30-35.
- Diewert, W.E. (1996b), "Price and Volume Measures in the System of National Accounts", pp. 237-285 in *The New System of National Accounts* J.W. Kendrick (ed.), Kluwer Academic Publishers, Boston.
- Diewert, W.E. (1997), "Commentary", *Federal Reserve Bank of St. Louis Review* 79:3, 127-137.
- Diewert, W.E. (1999), "Index Number Approaches to Seasonal Adjustment", *Macroeconomic Dynamics* 3, 48-68.
- Diewert, W.E. and K.J. Fox (1998), "The Measurement of Inflation after Tax Reform", *Economics Letters* 61, 279-284.
- Diewert, W.E. and D.A. Lawrence (1999), "Progress in Measuring the Price and Quantity of Capital", Discussion Paper 99-17, Department of Economics, University of British Columbia, Vancouver, Canada, V6T 1Z1, June.
- Edgeworth, F.Y. (1901), "Mr. Walsh on the Measurement of General Exchange Value", *The Economic Journal* 11, 404-416.
- Edgeworth, F.Y. (1923), "The Doctrine of Index Numbers According to Mr. Correa Walsh", *The Economic Journal* 11, 343-351.
- Edgeworth, F.Y. (1925), *Papers Relating to Political Economy*, Volume 1, New York: Burt Franklin.
- Eichhorn, W. (1978), *Functional Equations in Economics*, Reading, MA: Addison-Wesley Publishing Company.
- Eurostat (European Statistical Agency), World Bank, International Monetary Fund, Organization for Economic Cooperation and Development, and United Nations (1993), *System of National Accounts 1993*, United Nations, New York.
- Fisher, F. and K. Shell (1972), *The Economic Theory of Price Indexes*, New York: Academic Press.
- Fisher, I. (1911), *The Purchasing Power of Money*, London: Macmillan.
- Fisher, I. (1921), "The Best Form of Index Number", *Journal of the American Statistical Association* 17, 533-537.
- Fisher, I. (1922), *The Making of Index Numbers*, Houghton-Mifflin, Boston.
- Frisch, R. (1936), "Annual Survey of General Economic Theory: the problem of Index Numbers", *Econometrica* 4, 1-38.
- Greenlees, J. S. (1998), "Random Errors and Superlative Indexes", unpublished manuscript, Bureau of Labor Statistics, Washington, D.C., August 12.
- Harper, M.J., E.R. Berndt and D.O. Wood (1989), "Rates of Return and Capital Aggregation Using Alternative Rental Prices", pp. 331-372 in *Technology and*

- Capital Formation*, D.W. Jorgenson and R. Landau (eds.), The MIT Press, Cambridge, MA.
- Hicks, J.R. (1940), "The Valuation of the Social Income", *Economica* 7, 105-140.
- Hicks, J.R. (1941-42), "Consumers' Surplus and Index Numbers", *The Review of Economic Studies* 9, 126-137.
- Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarendon Press.
- Hill, T.P. (1988), "Recent Developments in Index Number Theory and Practice", *OECD Economic Studies* 10, 123-148.
- Hill, T.P. (1993), "Price and Volume Measures", pp. 379-406 in *System of National Accounts 1993*, Eurostat, IMF, OECD, UN and World Bank, Luxembourg, Washington, D.C., Paris, New York, and Washington, D.C.
- Hill, T.P. (1999a), "COL Indexes and Inflation Indexes", paper tabled at the 5th Meeting of the Ottawa Group on Price Indices, Reykjavik, Iceland, August 25-27, 1999.
- Hill, T.P. (1999b), "Capital Stocks and Flows", unpublished paper.
- Hulten, C.R. (1990), "The Measurement of Capital", pp. 119-158 in *Fifty Years of Economic Measurement*, E.R. Berndt and J.D. Triplett (eds.), Chicago: the University of Chicago Press.
- Jensen, J.L.W.V. (1906), "Sur les fonctions convexes et les inégalités entre les valeurs moyennes", *Acta Mathematica* 30, 175-193.
- Keynes, J.M. (1930), *A Treatise on Money in Two Volumes: 1: The Pure Theory of Money*, London: Macmillan.
- Kohli, U. (1978), "A Gross National Product Function and the Derived Demand for imports and Supply of Exports", *Canadian Journal of Economics* 11, 167-182.
- Kohli, U. (1991), *Technology, Duality and Foreign Trade: The GNP Function Approach to Modeling Imports and Exports*, Ann Arbor: University of Michigan Press.
- Konüs, A.A. (1924), "The Problem of the True Index of the Cost of Living", translated in *Econometrica* 7, (1939), 10-29.
- Knibbs, Sir G.H. (1924), "The Nature of an Unequivocal Price Index and Quantity Index", *Journal of the American Statistical Association* 19, 42-60 and 196-205.
- Leontief, W. (1936), "Composite Commodities and the Problem of Index Numbers", *Econometrica* 4, 39-59.
- Lloyd, P.J. (1975), "Substitution Effects and Biases in Nontrue Price Indices", *American Economic Review* 65, 301-313.
- Malmquist, S. (1953), "Index Numbers and Indifference Surfaces", *Trabajos de Estadística* 4, 209-242.
- Marshall, A. (1887), "Remedies for Fluctuations of General Prices", *Contemporary Review* 51, 355-375.
- Marshall, A. (1898), *Principles of Economics*, Fourth Edition, London: The Macmillan Co.
- Matheson, E. (1910), *The Depreciation of Factories and their Valuation*, Fourth Edition, London: E. & F.N. Spon.
- Montgomery, J.K. (1937), *The Mathematical Problem of the Price Index*, London: P.S. King & Son.
- Moulton, B.R. (1996), "Constant Elasticity Cost-of-Living Index in Share Relative Form", Bureau of Labor Statistics, Washington D.C., December.

- Nordhaus, W.D. (1997), "Do Real Output and Real Wage Measures Capture Reality? The History of Lighting Suggests Not", pp. 29-66 in *The Economics of New Goods*, T.F. Bresnahan and R.J. Gordon (eds.), University of Chicago Press, Chicago.
- Paul, Catherine J. Morrison (1999), *Cost Structure and the Measurement of Economic Performance*, Boston: Kluwer Academic Publishers.
- Pollak, R.A. (1980), "Group Cost-of-Living Indexes", *American Economic Review* 70, 273-278.
- Pollak, R.A. (1981), "The Social Cost-of-Living Index", *Journal of Public Economics* 15, 311-336.
- Pollak, R.A. (1983), "The Theory of the Cost-of-Living Index", pp. 87-161 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada; reprinted as pp. 3-52 in R.A. Pollak, *The Theory of the Cost-of-Living Index*, Oxford: Oxford University Press, 1989; also reprinted as pp. 5-77 in *Price Level Measurement*, W.E. Diewert (ed.), Amsterdam: North-Holland, 1990.
- Pollak, R.A. (1989), "The Treatment of the Environment in the Cost-of-Living Index", pp. 181-185 in R.A. Pollak, *The Theory of the Cost-of-Living Index*, Oxford: Oxford University Press.
- Prais, S.J. (1959), "Whose Cost of Living?", *The Review of Economic Studies* 26, 126-134.
- Samuelson, P.A. (1974), "Complementarity—An Essay on the 40th Anniversary of the Hicks-Allen Revolution in Demand Theory", *Journal of Economic Literature* 12, 1255-1289.
- Samuelson, P.A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64, 566-593.
- Selvanathan, E.A. and D.S. Prasada Rao (1994), *Index Numbers: A Stochastic Approach*, Ann Arbor: The University of Michigan Press.
- Shapiro, M.D. and D.W. Wilcox (1997), "Alternative Strategies for Aggregating Prices in the CPI", *Federal Reserve Bank of St. Louis Review* 79:3, 113-125.
- Sidgwick, H. (1883), *The Principles of Political Economy*, London: Macmillan.
- Romer, P. (1994), "New Goods, Old Theory and the Welfare Costs of Trade Restrictions", *Journal of Development Economics* 43, 5-38.
- Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland.
- Törnqvist, Leo (1936), "The Bank of Finland's Consumption Price Index," *Bank of Finland Monthly Bulletin* 10: 1-8.
- Törnqvist, L. and E. Törnqvist (1937), "Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.
- Triplett, J.E. (1983), "Escalation Measures: What is the Answer? What is the Question?", pp. 457-482 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.
- Triplett, J.E. (1999), "Should the Cost-of-Living Index Provide the Conceptual Framework for a Consumer Price Index?", paper presented at the 5th Meeting of the Ottawa Group on Price Indices, Reykjavik, Iceland, August 25-27, 1999.

- Turvey, R. (1979), "The Treatment of Seasonal Items in Consumer Price Indexes", *Bulletin of Labour Statistics* 4th Quarter, International Labour Office, Geneva, 13-33.
- Turvey, R. (1999), "True Cost of Living Indexes", paper presented at the 5th Meeting of the Ottawa Group on Price Indices, Reykjavik, Iceland, August 25-27, 1999.
- Vartia, Y.O. (1976), *Relative Changes and Index Numbers*, Helsinki: Research Institute of the Finnish Economy.
- Vogt, A. (1980), "Der Zeit und der Faktorkehrtest als 'Finders of Tests'", *Statistische Hefte* 21, 66-71.
- Wales, T.J. and A.D. Woodland (1976), "Estimation of Household Utility Functions and Labor Supply Response", *International Economic Review* 17, 397-410.
- Wales, T.J. and A.D. Woodland (1979), "Labour Supply and Progressive Taxes", *The Review of Economic Studies* 46, 83-95.
- Walras, L. (1954), *Elements of Pure Economics*, translated by W. Jaffe (first published in 1874), London: George Allen and Unwin.
- Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.
- Walsh, C.M. (1921), *The Problem of Estimation*, London: P.S. King & Son.
- Watson, W. (1999), "Labour Day, From the Front Porch", *Financial Post*, Toronto, Canada, September 8.
- Woodland, A.D. (1982), *International Trade and Resource Allocation*, Amsterdam: North-Holland.
- Woolford, K. (1999), "Measuring Inflation: A Framework Based on Domestic Final Purchases", paper presented at the Conference on the Measurement of Inflation, Cardiff University, Conference Organisers: Mick Silver and David Fenwick, August 31-September 1.
- Wynne, M.A. (1997), "Commentary", *Federal Reserve Bank of St. Louis Review* 79:3, 161-167.